HYPER VERSION

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A non-rational CFT with $c = 1$ as a limit of minimal models

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ABSTRACT: We investigate the limit of minimal model conformal field theories where the central charge approaches one. We conjecture that this limit is described by a non-rational CFT of central charge one. The limiting theory is different from the free boson but bears some resemblance to Liouville theory. Explicit expressions for the three point functions of bulk fields are presented, as well as a set of conformal boundary states. We provide analytic and numerical arguments in support of the claim that this data forms a consistent CFT.

KEYWORDS: Conformal Field Models in String Theory, Conformal and W Symmetry.

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1. Introduction

The unitary minimal models were discovered in [2]. They are two dimensional conformal field theories with the special property that they only contain a finite number of primary fields with respect to the Virasoro algebra. The possible partition functions for these models were classified in [3], and there is one unitary minimal model M_p for each integer $p \geq 2$ with a diagonal modular invariant. (Throughout this paper we will only consider theories with diagonal modular invariant.) M_p has central charge

$$c = 1 - \frac{6}{p(p+1)}\,, \tag{1.1}$$

The first non-trivial minimal model, M_3, with central charge $c = 1/2$, corresponds to the critical Ising model. At the other end, for $p \to \infty$, we see that the central charge approaches the limiting value $c = 1$.

Each of the minimal models has a set of conformal boundary conditions and in [16] it was shown how a particular limit of these boundary conditions is useful in clarifying the space of boundary renormalisation group flows in the minimal models. In that paper, only the boundary theories were considered, and it was found that at the limit point $c = 1$ there is an infinite discrete set of fundamental conformal boundary conditions, labelled by positive integers a. All other conformal boundary conditions obtained in the $c \to 1$ limit can be expressed as superpositions of these fundamental ones.

It appears that none of the known $c = 1$ theories has such a set of boundary conditions, and so it is natural to ask if it is possible to define a 'limiting minimal model'

$$M_\infty = \lim_{p\to\infty} M_p\,, \tag{1.2}$$

such that the boundary conditions of M_∞ are those discussed in [16]. It seems probable that one may take the limit $p \to \infty$ of the minimal models in more than one way [4, 23], but we shall not be concerned with such issues here and concentrate on finding a bulk field theory associated to the boundary conditions discussed in [16].

Each minimal model M_p has a finite number of Virasoro primary bulk fields, and requiring that the boundary conditions remain a discrete set each with a finite number of boundary fields implies that the bulk spectrum becomes continuous in the limit $p \to \infty$. M_∞ thus contains a continuum of primary bulk fields of conformal weight h. For later convenience we shall parametrise these fields in terms of $x = 2\sqrt{h}$ (so that $h_x = x^2/4$) and denote them by $\phi_x(\zeta, \bar{\zeta})$ (n.b. $h = \bar{h}$ for all primary fields in M_∞).

Curiously, the resulting bulk theory for M_∞ is not that of a free boson. In the case of the free boson, conservation of the U(1) charge strongly constrains the operator product expansion of two vertex operators. If the vertex operator V_x has charge x, then only the one vertex operator with correct charge can appear on the right-hand side of the OPE:

$$V_x(\zeta, \bar{\zeta})\, V_y(0,0) = |\zeta|^{xy} \left(V_{x+y}(0,0) + \text{descendants} \right), \tag{1.3}$$

For M_∞ however there is no corresponding conserved charge and one finds a continuum of primary fields when taking the OPE:

$$\phi_x(\zeta, \bar{\zeta})\, \phi_y(0,0) = \int |\zeta|^{2h_z - 2h_x - 2h_y}\, c(x,y,z) \left(\phi_z(0,0) + \text{descendants} \right) \mathrm{d}z\,, \tag{1.4}$$

In this respect M_∞ is rather similar to Liouville theory [8, 30, 25], and indeed the relation of this theory to Liouville theory is a major unanswered question.

A conformal field theory is rational (with respect to a chiral algebra) if it contains a finite number of primary fields with respect to that algebra, and quasi-rational (a weaker condition) if the operator product expansion of any two primary fields contains contributions from only a finite number of primary fields. A free boson is not rational with respect to either the Virasoro algebra or the larger U(1)-current algebra, but is quasi-rational. M_∞ on the other hand is, just like Liouville theory, neither rational nor quasi-rational.

The main result of this paper is the explicit formula for the M_∞ structure constants $c(x,y,z)$ appearing in the OPE (1.4). This is given in section 2.1. We are able to test these structure constants by a check of crossing symmetry of the bulk four-point functions, both analytically (in the special case when the $c = 1$ chiral blocks are known) and numerically (in the general case). We also propose a set of boundary states which correspond to the discrete set of $c = 1$ boundary conditions found in [16].

The paper is divided in three parts. First the $c = 1$ theory M_∞ is presented, in terms of its two- and three-point functions on the complex plane, and in terms of boundary states and one point functions on the unit disc. Next, the correlators of M_∞ are worked out in terms of conformal blocks and numerical and analytic tests of crossing symmetry are described. The final section deals with the derivation of the M_∞ quantities presented before.

2. Properties of M_∞

In this section we describe in some detail the conjectured $c = 1$ theory, which was denoted by M_∞ in the introduction. All expressions given here have been obtained as the limits of the corresponding quantities in minimal models. For clarity of presentation, the somewhat technical calculations of these limits have been shifted to section 4.

2.1 M_∞ on the complex plane

For a unitary conformal field theory on the full complex plane all correlators are specified once the two- and three-point functions of the Virasoro primary bulk fields are known (how this works in the example of the four point function is shown in some detail in section 3). Conformal invariance implies that all fields except for the identity will have vanishing one point functions on the complex plane. The functional dependence of two- and three-point functions is also fixed by conformal invariance, at least up to a constant. In the two point functions we can choose this constant at our convenience by redefining the primary fields. The only non trivial input specifying the theory on the complex plane is thus the primary fields which are present and the values of the numerical constants appearing in the three point functions.

We propose that the Hilbert space of M_∞ contains highest weight states $|x\rangle$ of conformal weights $h_x = \bar{h}_x = x^2/4$ for all positive x with the exception of the positive integers i.e. the set

$$\mathbb{S} = \mathbb{R}_{>0} - \mathbb{Z}_{>0}\,, \tag{2.1}$$

Corresponding to these states there are bulk primary fields

$$\phi_x(\zeta, \bar{\zeta})\,, \tag{2.2}$$

of the same conformal weights. Note that the fields taken out of $\mathbb{S}$ correspond exactly to those values of h for which the $c = 1$ Virasoro highest weight representations have null vectors (see section D).

Among the weights absent from $\mathbb{S}$ is $h = 0$, corresponding to the identity operator and its descendants (including $T(z)$ and $\bar{T}(\bar{z})$). Naturally we do not want to exclude these from the spectrum of fields, but it does appear that there is no actual vacuum state $|0\rangle$ in the theory.[1] In this case, the reconstruction of correlators involving the identity field and its descendants is possible using the conformal Ward identities (expressing the commutation relations of the Virasoro algebra with primary fields). However, we are also able to show that the operator $\lim_{x\to 0} \frac{d}{dx}\phi_x(\zeta, \bar{\zeta})$ behaves exactly as the identity field, and so in this way the identity can be constructed as a limit of the fields in $\mathbb{S}$. Similarly, while it may be possible to construct fields for the other integer values of x in terms of the ϕ_x, there are no contributions from states with these weights to any correlation function of fields in $\mathbb{S}$, and so we have not detected their presence in the limit theory we define. We discuss this further in sections 2.3 and 3.2.1.

[1] In equation (2.23) we define states $|0\rangle$ and $\langle 0|$ as limits of states in $\mathbb{S}$, but they are not normalisable.

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We take the fields ϕ_x to have the following two- and three-point functions on the complex plane:[2]

$$\begin{aligned}\langle \phi_x(\zeta_1,\bar{\zeta}_1)\,\phi_y(\zeta_2,\bar{\zeta}_2)\rangle &= \delta(x-y)\cdot|\zeta_{12}|^{-x^2}\,,\\ \langle \phi_x(\zeta_1,\bar{\zeta}_1)\,\phi_y(\zeta_2,\bar{\zeta}_2)\,\phi_z(\zeta_3,\bar{\zeta}_3)\rangle &= c(x,y,z)\cdot|\zeta_{12}|^{(z^2-x^2-y^2)/2}\,\times\\ &\quad\times|\zeta_{13}|^{(y^2-x^2-z^2)/2}\,|\zeta_{23}|^{(x^2-y^2-z^2)/2}\,,\end{aligned} \tag{2.3}$$

where $\zeta_{ij}=\zeta_i-\zeta_j$ and the structure constants $c(x,y,z)$ are of the form

$$c(x,y,z) = P(x,y,z)\cdot\exp(Q(x,y,z))\,, \tag{2.4}$$

$P(x,y,z)$ is a step function, defined as follows: we denote by $[x]$ the largest integer less than or equal to x and define $f_x = x-[x]$ to be the fractional part of x. Then $P(x,y,z)$ takes the form

$$P(x,y,z)=\begin{cases}\frac{1}{2}: & \big([x]{+}[y]{+}[z]\text{ even, and }|f_x{-}f_y|<f_z<\min(f_x{+}f_y,2{-}f_x{-}f_y)\big)\ \text{ or}\\ & \big([x]{+}[y]{+}[z]\text{ odd, and }|f_x{-}f_y|<1{-}f_z<\min(f_x{+}f_y,2{-}f_x{-}f_y)\big)\\ 0: & \text{otherwise}\end{cases} \tag{2.5}$$

One can verify that P is symmetric in all indices and periodic $P(x,y,z){=}P(x,y,z{+}2)$. The function $Q(x,y,z)$ in (2.4) is given by

$$Q(x,y,z)=\int_0^1\frac{\mathrm{d}\beta}{(-\ln\beta)\cdot(1{-}\beta)^2}\cdot\left\{2+\sum_{\varepsilon=\pm1}\left(\beta^{\varepsilon x}{+}\beta^{\varepsilon y}{+}\beta^{\varepsilon z}\right)-\sum_{\varepsilon_x,\varepsilon_y,\varepsilon_z=\pm1}\beta^{(\varepsilon_x x+\varepsilon_y y+\varepsilon_z z)/2}\right\}, \tag{2.6}$$

The integral converges when $x,y,z<1$ and $x{+}y{+}z<2$. Recall that by definition we have $x,y,z>0$. For larger values the integral (2.6) has to be obtained by analytic continuation in x,y,z. This makes the formula look more cumbersome, but for numerical studies it is helpful to have the analytic continuation done explicitly. The result is given in appendix B. The explicit formula also implies that

$$c(x,y,z)\geq 0\,, \tag{2.7}$$

From the formulae (2.5) and (2.6) we see that the structure constants $c(x,y,z)$ are symmetric in all three indices. In figure 1 the structure constants have been plotted as functions of z for two choices of the values x,y. We note here that the structure constants simplify greatly if two of x,y and z are half-integer, in which case $P=1/2$ and (if x and y are half-integer) $\exp(Q)=2^{-z^2}\cdot q(z)$ where $q(z)$ is a polynomial, the simplest case being $c(1/2,1/2,z)=2^{-1-z^2}$. This is discussed in appendix C.

[2] We consider the relation of these correlation functions to the operator product algebra in section 2.4.

The data given so far allows us (in principle) to work out all correlators on the complex plane involving fields ϕ_x with $x \in \mathbb{S}$. In section (2.3) below we discuss consistency of the data presented above and point out some further open questions.

Before that we would like to define boundary states for M_∞. These enable us to work out correlators of bulk fields on the unit disc and to compute cylinder partition functions and check that these have the properties that we demanded in [16].

2.2 The boundary conditions of M_∞

M_∞ was defined to be consistent with the boundary field theories considered in [16], so we shall first recall briefly these results.

In a minimal model the boundary conditions are in one-one correspondence with the conformal weights, and so can be labelled by a pair of integers (r, s). In the limit $c \to 1$ these can all be expressed as suitable superpositions of a fundamental set of boundary conditions $\hat{a}$ which are the limits of the boundary conditions $(a, 1)$. These boundary conditions and their field content at $c = 1$ are discussed in detail in [16]. Note that we do not want to imply that the set $a \in \mathbb{Z}_{>0}$ exhausts all conformal boundary conditions of M_∞. There could well be others which can not, or only in a less direct way, be obtained from the minimal model limit. We have nothing to say about those.

The boundary conditions $\hat{a}$ were defined so that the Hilbert space on the strip with boundary conditions $\hat{a}$ and $\hat{1}$ is a single irreducible highest-weight representation $L(h_{a-1})$ of the Virasoro algebra of weight $h_{a-1} = (a-1)^2/4$. This in turn means that the partition function on the strip $Z_{\hat{a}\hat{1}}$ is a single Virasoro character. If the strip is of width r and length ℓ, then

$$Z_{\hat{a}\hat{1}}(r, \ell) = \mathrm{Tr}_{L(h_{a-1})}\left(e^{-\pi\ell/r(L_0 - c/24)}\right) = \chi_{(a-1)}(q)\,, \tag{2.8}$$

where $q = \exp(-\pi\ell/r)$ and $\chi_x(q)$ is the character of the irreducible representations of weight h_x (given in appendix D). Using the modular transformation in appendix D,

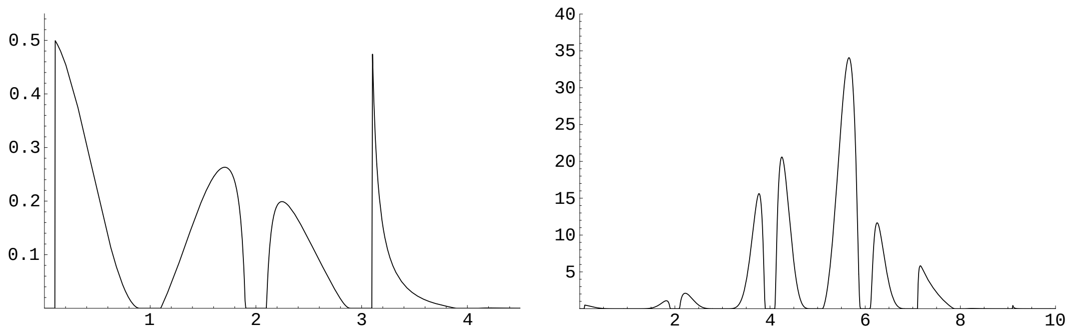

Figure 1: Two example plots of the M_∞ structure constants. The left plot shows $c(1.5, 1.6, z)$ as a function of z, the right plot shows $c(4.5, 4.6, z)$.

we can express the partition function in terms of characters of the crossed channel (the 'closed string channel') as

$$Z_{\hat{a}\hat{1}}(r,\ell) = 2^{3/2} \int_0^\infty \sin(a\pi x)\, \sin(\pi x)\, \chi_x(\tilde{q})\, \mathrm{d}x\,, \tag{2.9}$$

where $\tilde{q} = \exp(-4\pi r/\ell)$.

Note that the degenerate representations with x integer do not contribute to this partition function. This means that we can recover (2.9) in the boundary state formalism with boundary states that only include states in $\mathbb{S}$. We propose that the boundary states are given by

$$|\hat{a}\rangle = 2^{3/4} \int_0^\infty (-1)^{[x]} \sin(a\pi x)\, |x\rangle\!\rangle\, \mathrm{d}x\,, \tag{2.10}$$

The factor $(-1)^{[x]}$, where $[x]$ is again the integer part of x, is linked to our choice of normalisation of the bulk fields. $|x\rangle\!\rangle$ and $\langle\!\langle x|$ are Ishibashi states [17] corresponding to the bulk fields of weight h_x and which satisfy

$$\langle\!\langle x|\tilde{q}^{-\frac{1}{2}(L_0+\bar{L}_0-1/12)}|y\rangle\!\rangle = \delta(x-y)\, \chi_x(\tilde{q})\,, \tag{2.11}$$

As a check we can calculate the partition function on a cylinder with boundary conditions $\hat{a}$ and $\hat{b}$ on the two ends as

$$Z_{\hat{a},\hat{b}} = \langle\hat{a}|e^{-\frac{2\pi r}{\ell}(L_0+\bar{L}_0-\frac{1}{12})}|\hat{b}\rangle = \int_0^\infty \mathrm{d}x\, 2^{3/2}\cdot\sin(\pi a x)\cdot\sin(\pi b x)\cdot\chi_x(\tilde{q}) = \sum_{k\in a\otimes b} \chi_{k-1}(q)\,, \tag{2.12}$$

Here $a\otimes b$ stands for the su(2)-fusion product and is a short hand for the range of the sum (e.g. for $(2)\otimes(2) = (1)+(3)$ the sum runs over $k=1,3$). The result (2.12) agrees with that in [16].

The final ingredient one needs to be able to compute all amplitudes on the unit disk are the one-point functions of bulk fields, which one can read off directly from the boundary state (2.10):

$$\langle\phi_x(0,0)\rangle^{\hat{a}}_{\text{disc}} = \langle\hat{a}|x\rangle = 2^{3/4}(-1)^{[x]}\sin(\pi a x)\,, \tag{2.13}$$

To compute normalised expectation values, one needs the unit disc partition function $Z^{\hat{a}}$ and the result from the minimal model limit is

$$Z^{\hat{a}} = 2^{3/4}\pi a\,, \tag{2.14}$$

2.3 Unknown properties of M_∞

Above we have presented expressions for correlators as they are obtained from the $c \to 1$ limit of minimal models. That the expressions are found in a limiting procedure from well defined conformal field theories is a reason to hope, but by no means a proof, that M_∞ is a consistent theory by itself.

The first question is whether the correlation functions are crossing symmetric, i.e. whether they are independent of how one chooses to expand them in three point functions and intermediate states. In section 3 numerical and analytic evidence is presented that the four point function is indeed crossing symmetric.

The second question is whether $\mathbb{S}$ is really the full spectrum of the theory. One check is again crossing symmetry of the correlators, which fails if not all states are included in the intermediate channels. On the other hand it might be possible to add in fields and/or states to the full theory which simply do not contribute as intermediate channels in the four-point functions we consider.

At the present stage we do not have a definite answer to this question and it remains for future research. One may however speculate that it is possible to extend the spectrum of fields to all non negative real numbers $\mathbb{R}_{\geq 0}$. As an example let us consider the possibility to add a field of weight $h = 0$ to the theory. For very small x, the structure constants $c(x, y, z)$ in (2.4) behave as

$$c(x, y, z) \sim \frac{1}{2}\theta(x - y + z)\theta(x + y - z)\,; \qquad x \ll 1\,, \tag{2.15}$$

The function $\theta(x)$ is taken to be zero for $x \leq 0$ and one for $x > 0$. Using this one finds that $\lim_{x\to 0}\langle \phi_x\, \phi_y\, \phi_z\rangle = \delta_{y,z}$. The δ-symbol here is meant as a discrete (Kronecker) delta, not as a Dirac delta distribution. So $\delta_{y,z}$ is a function which is one if $y = z$ and zero otherwise. In particular we do not recover the two point function (2.3) in this limit. Thus $\lim_{x\to 0}\phi_x$ is not a good candidate for our tentative field of weight zero. However, since

$$\tfrac{d}{dx}c(x, y, z) \sim \frac{1}{2}\big(\delta(x - y + z) + \delta(x + y - z)\big)\,; \qquad x \ll y, z\,, \tag{2.16}$$

the field $\hat{\phi}_0(\zeta, \bar{\zeta}) = \lim_{x\to 0}\frac{d}{dx}\phi_x(\zeta, \bar{\zeta})$ has the property

$$\langle \hat{\phi}_0\, \phi_y\, \phi_z\rangle = \delta(y - z)\,, \tag{2.17}$$

If we repeat the above calculation with the alternative definition $\hat{\phi}_0(\zeta, \bar{\zeta}) = \lim_{x\to 0}\frac{1}{x}\phi_x(\zeta, \bar{\zeta})$ and are led to the same answer. This can be understood since $\lim_{x\to 0}\phi_x = 0$ inside correlators, at least in a distributional sense (it can be nonvanishing on a measure zero set, as we have seen above). Further, using (2.13) and (2.14) we find that the normalised one point function on the unit disc takes the form

$$\frac{\langle \hat{\phi}_0(0,0)\rangle^{\hat{a}}_{\text{disc}}}{Z^{\hat{a}}} = \lim_{x\to 0}\frac{d}{dx}\,\frac{\sin(\pi a x)}{\pi a} = 1\,, \tag{2.18}$$

So in these two examples the field $\hat{\phi}_0$ behaves like an identity field. One can now speculate that $\hat{\phi}_0$ is indeed a sensible field to consider and that similar constructions are possible for the other missing points in $\mathbb{S}$. However, our checks on crossing suggest very strongly that these fields do not appear in the operator product expansion of the fields in $\mathbb{S}$, as we discuss in section 3.

A third question concerns the connection of M_∞ to Liouville theory (see e.g. [25] for a recent review). The expressions for the bulk structure constants are very similar to the ones of Liouville theory as given in [8, 30]. In appendix A this correspondence is made more precise. Furthermore, in Liouville theory one equally finds a discrete series of boundary conditions, whose field content consist solely of degenerate Virasoro representations [31]. It would be interesting to know whether M_∞ can be obtained from Liouville theory as a limit $c \to 1$ from above and whether methods based on continuous sets of representations as presented in [21, 25] could be used to establish consistency of M_∞.

A fourth point open for speculation is whether the set of conformal boundary conditions $\hat{a}$, one for each positive integer, is complete, i.e. if any conformal boundary condition of M_∞ can be rewritten as a superposition of the boundary conditions $\hat{a}$. Again we cannot make any definite statement, and this problem remains open for future work. However two arguments hint that there should be more boundary conditions. Firstly, from a RCFT point of view, there should be one fundamental conformal boundary condition for each diagonal field in the bulk spectrum [6, 1]. In M_∞ the bulk spectrum is continuous, so one might expect a continuum of boundary conditions. Secondly, in Liouville theory one equally finds a continuum of conformal boundary conditions [13].

2.4 Operator product expansion and the vacuum state

In (2.3) we have defined M_∞ in the bulk via its two and three point function. We would like to understand the constants appearing in the three point function as coefficients in the operator product expansion of two primary fields. The most obvious OPE one could write down would be

$$\phi_x(\zeta_1,\bar\zeta_1)\phi_y(\zeta_2,\bar\zeta_2) = \int_{\mathbb{S}} \mathrm{d}z\; c(x,y,z)\; |\zeta_1-\zeta_2|^{2h_z-2h_x-2h_y}\; \big(\phi_z(\zeta_2,\bar\zeta_2) + \text{descendants}\big)\,, \tag{2.19}$$

This expression however causes an immediate problem when inserted in the two point function (2.3). In this case we should recover the δ-function, but since all fields in the spectrum $\mathbb{S}$ have nonzero conformal weight, the r.h.s. of the OPE (2.19) vanishes identically.

A way around this problem can be found if one understands the states $|x\rangle$ and the corresponding fields $\phi_x(\zeta,\bar\zeta)$ in a distributional sense. To this end let us define states and fields with smeared out conformal weight (rather than position) as

$$|f\rangle = \int_{\mathbb{S}} \mathrm{d}x\, f(x)\,|x\rangle \qquad \text{and} \qquad \phi_f(\zeta,\bar\zeta) = \int_{\mathbb{S}} \mathrm{d}x\, f(x)\,\phi_x(\zeta,\bar\zeta)\,, \tag{2.20}$$

The smeared states have the property $\langle x|f\rangle = f(x)$ and the OPE of the smeared fields takes the form

$$\phi_f(\zeta_1,\bar\zeta_1)\phi_g(\zeta_2,\bar\zeta_2) = \phi_h(\zeta_2,\bar\zeta_2) + \text{descendants}\,, \tag{2.21}$$

where

$$h(z) = \int_{\mathbb{S}} \mathrm{d}x \int_{\mathbb{S}} \mathrm{d}y \; c(x,y,z)\; f(x)\, g(y)\; |\zeta_1 - \zeta_2|^{2h_z - 2h_x - 2h_y}\,, \tag{2.22}$$

In order to determine the set of test functions f we want to allow, let us define a linear functional $\langle 0|$ which will correspond to the out vacuum. The expression for $\hat{\phi}_0$ in (2.17) motivates the definitions

$$\langle 0| = \lim_{x\to 0} \frac{1}{x} \langle x|\,, \qquad \text{and} \qquad |0\rangle = \lim_{x\to 0} \frac{1}{x} |x\rangle\,, \tag{2.23}$$

As test functions we allow functions $f(x)$ such that $|f\rangle$ is in the domain of $\langle 0|$, i.e.

$$\langle 0|f\rangle = \lim_{x\to 0} \frac{f(x)}{x} \tag{2.24}$$

has to be well defined. This fixes the behaviour of f close to zero. Additionally we want f to be square integrable and continuous everywhere except for jumps which happen at a finite distance from each other (to allow for the factor P in (2.4)).

Similarly we can ask what properties $f(x)$ must have such that the operator $\phi_f(\zeta,\bar{\zeta})$ maps $|0\rangle$ into the domain of $\langle 0|$. Using the approximation (2.15) for the structure constants we find, for $|\zeta| > 0$,

$$\langle 0|\phi_f(\zeta,\bar{\zeta})|0\rangle = \lim_{x\to 0} \lim_{y\to 0} \frac{\langle x|\phi_f(\zeta,\bar{\zeta})|y\rangle}{xy} = \lim_{x\to 0} \frac{f(x)}{x}\,. \tag{2.25}$$

This is the same condition as (2.24) for the inner product $\langle 0|f\rangle$, consistent with the idea that the state field correspondence will manifest itself in the form $\lim_{\zeta\to 0} \phi_f(\zeta,\bar{\zeta})|0\rangle = |f\rangle$.

Let us see how the above distributional interpretation of fields and states in $\mathbb{S}$ resolves the problem encountered when inserting the OPE (2.19) into the two point function:

$$\langle 0|\phi_f(\zeta_1,\bar{\zeta}_1)\phi_g(\zeta_2,\bar{\zeta}_2)|0\rangle = \langle 0|\phi_h(\zeta_2,\bar{\zeta}_2)|0\rangle = \lim_{z\to 0} \frac{h(z)}{z} = \int_{\mathbb{S}} f(x) g(x) |\zeta_1 - \zeta_2|^{-4h_x} \mathrm{d}x\,. \tag{2.26}$$

Here we used first the definition of the OPE (2.21), then the result for the one point function (2.25) and finally inserted the approximation (2.15) to take the $z \to 0$ limit of (2.22). The original two point function in (2.3) is exactly the formulation of this result in terms of distributions.

The treatment above is modelled closely after a similar situation occurring in Liouville theory. There the set of conformal weights of states in the spectrum does not have zero as a limit point. The sl(2)-invariant vacuum is thus even at a finite distance from the spectrum of states. As treated in detail in sections 4.6ff of [25], to define expressions like $\langle 0|f\rangle$ in Liouville theory, one has stronger requirements on the test function f, e.g. it should allow for an analytic continuation to zero.

3. Crossing symmetry

The most stringent consistency condition the proposed $c = 1$ theory M_∞ has to fulfill is that of crossing symmetry in its various incarnations. In this section it will be verified analytically and numerically in two examples: for four bulk fields on the full complex plane and for two bulk fields on the upper half plane.

In order to compute the respective correlators we will need the four point conformal blocks, from which the correlators entering the two examples can be constructed.

3.1 Conformal blocks

To motivate the construction of conformal blocks, it is helpful to think of a primary field $\phi_x(\zeta,\bar\zeta)$ as a linear operator on the Hilbert space of the theory, $\phi_x(\zeta,\bar\zeta) : \mathcal{H} \to \mathcal{H}$. The Hilbert space itself splits into representations of the Virasoro algebra, $\mathcal{H} = \int L(y)\otimes L(y)\mathrm{d}y$. We are deliberately vague on the meaning of the integral, as it is not important in what follows. Each $L(y)$ is an irreducible Virasoro highest weight module of weight $h = y^2/4$, as discussed in section D. We can now decompose the linear map $\phi_x(\zeta,\bar\zeta)$ into components $V^z_{xy}(\zeta)\otimes V^z_{xy}(\bar\zeta)$. The individual linear maps

$$V^z_{xy}(\zeta) : L(y) \to L(z) \tag{3.1}$$

are called *chiral vertex operators* (CVOs). In the choice of notation and presentation of the CVOs we follow [20]. By construction the CVOs are required to have the same commutation relations with the Virasoro generators as the primary field $\phi_x(\zeta,\bar\zeta)$ itself

$$[L_n, V^z_{xy}(\zeta)] = \zeta^n \cdot \left(h_x(n{+}1) + \zeta\frac{\mathrm{d}}{\mathrm{d}\zeta}\right) V^z_{xy}(\zeta)\,, \tag{3.2}$$

This defines the chiral vertex operator uniquely, up to an overall normalisation, in terms of its matrix elements. Using the commutation relations (3.2), a general matrix element

$$\langle z| L_{m_1}\dots L_{m_\ell}\; V^z_{xy}(\zeta)\; L_{-n_1}\dots L_{-n_k}|y\rangle \tag{3.3}$$

can then be reduced to a function of ζ times $\langle z|\; V^z_{xy}(\zeta)\;|y\rangle$. The latter has to be of the form

$$\langle z|\; V^z_{xy}(\zeta)\;|y\rangle = |V^z_{xy}| \cdot \zeta^{h_z-h_x-h_y}\,, \tag{3.4}$$

for some (not necessarily positive) number $|V^z_{xy}|$. This is the remaining freedom mentioned above. It follows that for the Virasoro algebra, the space of CVOs $V^z_{xy}(\zeta)$ is at most one dimensional. It can however have dimension zero. A simple example of this is given by $\langle z|\; V^z_{x0}(\zeta)\;|0\rangle$, where the null state $L_{-1}|0\rangle$ forces $\frac{\partial}{\partial\zeta}\langle z|\; V^z_{x0}(\zeta)\;|0\rangle = 0$ and thus $|V^z_{x0}| = 0$ if $x \neq z$.

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If we are allowed to choose $|V_{xy}^z|$ nonzero, we fix the normalisation of the chiral vertex operator by setting it to one.

Conformal blocks are given in terms of products of CVOs. For example the blocks needed for a four point correlator are $\langle t|V_{xp}^t(1)V_{yz}^p(\zeta)|z\rangle$ for all possible intermediate channels p. For better readability let us introduce the pictorial notation [20]

$$\langle t|V_{xp}^t(1)V_{yz}^p(\zeta)|z\rangle = \begin{array}{c|c|c} & x & y \\ t & p & z \\ \hline & 1 & \zeta \end{array}\,, \tag{3.5}$$

where $t, x, y, z, p \in \mathbb{S}$. Note that as opposed to the situation in minimal models, these conformal blocks do not obey any differential equations coming from null vectors. They can however still be computed order by order as a power series in ζ by working out the matrix elements of the CVOs.

As already noted, we can think of a bulk field as a linear operator made up of CVOs, i.e.

$$\phi_x(\zeta,\bar{\zeta}) = \int \mathrm{d}y \int \mathrm{d}z \; c(x,y,z)\; V_{xy}^z(\zeta) \otimes V_{xy}^z(\bar{\zeta})\,, \tag{3.6}$$

where again we understand the integral as a formal expression. Evaluating the bulk three point function in this formalism, one sees that the constants $c(x,y,z)$ are indeed the bulk structure constants appearing in (2.3). Building a bulk four point function from the operators (3.6), we note that it can be expressed as a bilinear combination of conformal blocks.

The computation of (3.5) as a power series in ζ via explicit evaluation of the CVOs becomes very slow for higher powers of ζ. However, there are at least two special situations in which the chiral blocks can be computed exactly, and an efficient algorithm due to Al. Zamolodchikov for numerical calculation in the remaining cases.

3.1.1 Exact conformal blocks

The first special case is when (3.5) can be interpreted as a product of free boson vertex operators, i.e. when the charges sum to zero $-t + x + p = 0$ and $-p + y + z = 0$. For $t, x, y, z, p \in \mathbb{S}$ one finds the Coulomb gas expression for the chiral four point function

$$\begin{array}{c|c|c} & x & y \\ t & p & z \\ \hline & 1 & \zeta \end{array} = (1-\zeta)^{xy/2} \cdot \zeta^{yz/2}\,, \tag{3.7}$$

The second special case is when (3.5) can be interpreted as a correlation function of descendants of weight $1/16$ spin fields for a $\mathbb{Z}_2$ twisted free boson (sometimes called the 'Ramond' sector) [27, 29]. The condition for this to be the case is that x, y, z and t are half integer. The simplest example is, for $p \in \mathbb{S}$,

$$\begin{array}{c|c|c} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & p & \frac{1}{2} \\ \hline & 1 & \zeta \end{array} = \frac{(16\,q)^{h_p}}{[\zeta(1-\zeta)]^{1/8}\,\theta_3(q)} \equiv G(\zeta)\,, \tag{3.8}$$

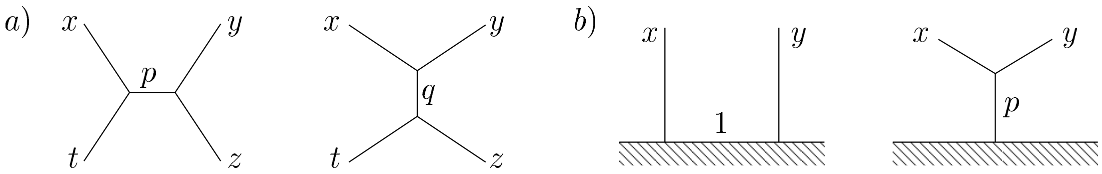


Figure 2: Crossing symmetry of a) the four point function in the complex plane and b) the two point function on the upper half plane.

where

$$q(\zeta) = \exp\Big(-\frac{\pi K(1-\zeta)}{K(\zeta)}\Big)\,, \quad K(\zeta) = \frac{1}{2}\int_0^1 \frac{\mathrm{d}t}{\sqrt{t(1-t)(1-\zeta t)}}\,, \quad \theta_3(q) = \sqrt{\frac{2K(\zeta)}{\pi}}\,, \tag{3.9}$$

All the other cases can be found in terms of differential operators acting on this function, for example the next two simplest cases are

$$\begin{array}{c|c|c} \tfrac{1}{2} & \tfrac{3}{2}\, p & \tfrac{3}{2}\;\tfrac{1}{2} \\ \hline & 1 & \zeta \end{array} = \frac{1}{2h_p}\,\zeta^{1/2}\,\left[2\frac{\mathrm{d}}{\mathrm{d}\zeta} + \frac{1}{4\zeta(1-\zeta)}\right] G(\zeta)\,, \tag{3.10}$$

$$\begin{array}{c|c|c} \tfrac{1}{2} & \tfrac{1}{2}\, p & \tfrac{3}{2}\;\tfrac{3}{2} \\ \hline & 1 & \zeta \end{array} = \frac{2}{4h_p - 1}\,(1-\zeta)^{1/2}\left[2\frac{\mathrm{d}}{\mathrm{d}\zeta} - \frac{1}{4\zeta(1-\zeta)}\right] G(\zeta)\,, \tag{3.11}$$

For general conformal blocks, we use Zamolodchikov's two recursive methods which are described in appendix E.

3.2 Four point function on the complex plane

We have now gathered the ingredients needed for checks of crossing symmetry. Let us start by considering the correlator of four bulk fields ϕ_t, ϕ_x, ϕ_y and ϕ_z on the complex plane. In figure 2a two different ways to insert a basis of intermediate states are depicted. In the first case one reduces the four point function to a two point function by taking the OPEs $\phi_t \times \phi_x$ and $\phi_y \times \phi_z$, in the second case one considers $\phi_t \times \phi_z$ and $\phi_x \times \phi_y$. This corresponds to two different ways of expressing the four point function in terms of conformal blocks, as worked out below.

Using conformal transformations, we can always move three of the four fields to the points $0, 1, \infty$, say. It is thus enough to consider the correlator

$$\langle t|\, \phi_x(1,1)\, \phi_y(\zeta,\bar\zeta)\, |z\rangle = \lim_{L\to\infty} L^{4h_t} \langle\, \phi_t(L,L)\, \phi_x(1,1)\, \phi_y(\zeta,\bar\zeta)\, \phi_z(0,0)\,\rangle\,, \tag{3.12}$$

We want to express (3.12) as a bilinear combination of conformal blocks of the form (3.5). This corresponds to inserting a basis of intermediate states between $\phi_t\phi_x$ and $\phi_y\phi_z$. The coefficients in front of each pair of conformal blocks are given by the bulk structure constants describing the coupling of $\phi_t\phi_x$ and $\phi_y\phi_z$ to the intermediate channel p

$$
\begin{aligned}
A(\zeta) &= \langle t|\ \phi_x(1,1)\,\phi_y(\zeta,\bar\zeta)\ |z\rangle \\
&= \int_{\mathbb{S}} \mathrm{d}p\ \ c(t,x,p)\cdot c(y,z,p)\ \underset{1\qquad \zeta}{\underline{t\ \ \big|^{x}\ p\ \big|^{y}\ \ z}}\ \underset{1\qquad \bar\zeta}{\underline{t\ \ \big|^{x}\ p\ \big|^{y}\ \ z}}\,,
\end{aligned}
\tag{3.13}
$$

The correlator (3.13) has to be invariant under conformal transformations. So if we apply for example the map $\xi \to 1-\xi$, we exchange the fields at zero and one, ζ gets mapped to $1-\zeta$ while the field ∞ remains fixed. In radial ordering we are now looking at the correlator $\langle t|\ \phi_z(1,1)\,\phi_y(1{-}\zeta,1{-}\bar\zeta)\ |x\rangle$. As before we can now insert a basis of intermediate states, this time between $\phi_t\phi_z$ and $\phi_y\phi_x$, and arrive at the following expression in terms of conformal blocks:

$$
\begin{aligned}
B(\zeta) &= \langle t|\ \phi_z(1,1)\,\phi_y(1{-}\zeta,1{-}\bar\zeta)\ |x\rangle \\
&= \int_{\mathbb{S}} \mathrm{d}q\ \ c(t,z,q)\cdot c(y,x,q)\ \underset{1\quad 1-\zeta}{\underline{t\ \ \big|^{z}\ q\ \big|^{y}\ \ x}}\ \underset{1\quad 1-\bar\zeta}{\underline{t\ \ \big|^{z}\ q\ \big|^{y}\ \ x}}\,,
\end{aligned}
\tag{3.14}
$$

The two function $A(\zeta)$ and $B(\zeta)$ now correspond to the two different expansions in figure 2*a*. Crossing symmetry demands that they are the same $A(\zeta) = B(\zeta)$.

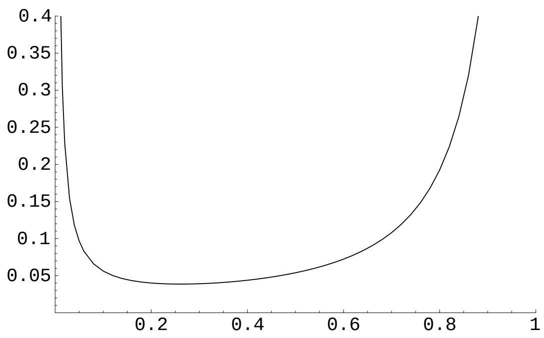
The first case we shall consider is the correlator of four fields of weight $h = \bar h = 1/16$, since in that case both the bulk structure constants simplify to $c(1/2,1/2,x) = 2^{-x^2-1}$ and we can use Zamolodchikov's exact formula for the conformal block (3.8). In this case we can find the four-point function exactly

$$
\begin{aligned}
\langle \tfrac{1}{2}|\,\phi_{\frac12}(1,1)\,\phi_{\frac12}(\zeta,\bar\zeta)\,|\tfrac{1}{2}\rangle &= \int_{\mathbb{S}} c(1/2,1/2,p)^2 \left| \underset{1\qquad \zeta}{\underline{\tfrac12\ \ \big|^{\frac12}\ p\ \big|^{\frac12}\ \tfrac12}} \right|^2 \mathrm{d}p \\
&= \int_0^\infty 2^{-2p^2-2}\,\frac{|\,16\,q\,|^{p^2/2}}{|\zeta(1-\zeta)|^{1/4}\,|\theta_3(q)|^2}\ \mathrm{d}p \\
&= \frac{\pi/2}{4\,|\zeta(1-\zeta)|^{1/4}\,\left|2\,\mathrm{Re}(K(1{-}\zeta)\,\overline{K(\zeta)})\right|^{1/2}}\,,
\end{aligned}
\tag{3.15}
$$

Since all the fields are the same, crossing symmetry in this case is just invariance under $\zeta \to 1-\zeta$, which is manifestly true.

We can also find all correlation functions of four fields with x, y, z and t all half-integral in the same way, by explicit integration using the exact conformal block and structure constants. For example, the correlation function of two fields $\phi_{1/2}$ and two fields $\phi_{3/2}$ can be decomposed in the two equivalent ways

$$
\begin{aligned}
\langle \tfrac{1}{2}|\,\phi_{\frac32}(1,1)\,\phi_{\frac32}(\zeta,\bar\zeta)\,|\tfrac{1}{2}\rangle &= \int_{\mathbb{S}} c\left(\frac12,\frac32,p\right)^2 \left| \underset{1\qquad \zeta}{\underline{\tfrac12\ \ \big|^{\frac32}\ p\ \big|^{\frac32}\ \tfrac12}} \right|^2 \mathrm{d}p \\
&= \int_{\mathbb{S}} c\left(\frac12,\frac12,p\right) c\left(\frac32,\frac32,p\right) \left| \underset{1\quad (1-\zeta)}{\underline{\tfrac12\ \ \big|^{\frac12}\ p\ \big|^{\frac32}\ \tfrac32}} \right|^2 \mathrm{d}p\,,
\end{aligned}
\tag{3.16}
$$



ζ	$A(\zeta)$	$\lvert A(\zeta)-B(\zeta)\rvert$
0.04	0.11802144 ...	0.00090869 ...
0.11	0.05285051 ...	0.00001312 ...
0.18	0.04134348 ...	0.00000041 ...
0.32	0.03964800 ...	0.00000023 ...
0.53	0.05830843 ...	0.00000014 ...
0.67	0.09444726 ...	0.00000007 ...
0.81	0.20750675 ...	0.00000136 ...
0.88	0.40080739 ...	0.00003664 ...
0.95	1.39461196 ...	0.00346871 ...

Figure 3: Example of crossing in the bulk with $t = 0.8$, $x = 1.1$, $y = 1.3$, $z = 0.7$. The data has been obtained by using the recursion relation (E.11) to level 12 and a numerical integration step width of 0.001. The plot shows the function $A(\zeta)$ as given in (3.13) against ζ.

both of which integrals can be evaluated exactly and both giving the same (less than illuminating) expression

$$\pi \frac{3\,\pi^2\,|K'|^2 - 8\,\pi\,\mathrm{Re}(\bar{K}K')\,\mathrm{Re}(E'K') + 16\,|E'|^2\,\left(\mathrm{Re}(\bar{K}K')\right)^2}{32\,\left|1-\zeta\right|^{9/4}\,|\zeta|^{5/4}\,|K'|^2\,\left|2\,\mathrm{Re}(\bar{K}K')\right|^{5/2}}\,, \tag{3.17}$$

where $K \equiv K(\zeta)$ and $E \equiv E(\zeta)$ are the standard complete Elliptic Integrals of the first and second kinds, $K' = K(1-\zeta), E' = E(1-\zeta)$, and the bar denotes complex conjugation.

For the remaining cases, we must use numerical tests only. We have used Zamolodchikov's recursive formulae for the conformal blocks and we find excellent agreement between the functions A and B. An example of the numerical evaluation is shown in figure 3.

3.2.1 Crossing and degenerate representations

We shall now argue that the degenerate representations, and in particular the identity representation, do not occur as separate intermediate channels in the bulk four-point correlation functions if we assume that the correlation functions are continuous in the weights of the fields. For general t, x, y, z, we have proposed the decomposition of the correlation function as

$$\langle t|\,\phi_x(1,1)\,\phi_y(\zeta,\bar{\zeta})\,|z\rangle = \int_{\mathbb{S}} c(t,x,p)\,c(y,z,p)\left|\begin{array}{ccccc} & x & & y & \\ t & \big| & p & \big| & z \\ \hline & 1 & & \zeta & \end{array}\right|^2 \mathrm{d}p\,, \tag{3.18}$$

The degenerate representations with p integer all contain singular vectors, and so the chiral blocks in which they can occur are limited by the requirement that these singular vectors decouple — for example, the 'vacuum' channel $p = 0$ is only allowed if $t = x$ and $y = z$.

If t, x, y and z satisfy such requirements, then we could *a priori* consider the more general expression

$$\langle t|\, \phi_x(1,1)\, \phi_y(\zeta,\bar\zeta)\, |z\rangle = \int_{\mathbb{S}} c(t,x,p)\, c(y,z,p) \left| \begin{array}{c|c|c} & x & y \\ t & p & z \\ \hline & 1 & \zeta \end{array} \right|^2 \mathrm{d}p \; + $$

$$+ \sum_{p=0}^{\infty} \tilde c(t,x,p)\, \tilde c(y,z,p) \left| \begin{array}{c|c|c} & x & y \\ t & p & z \\ \hline & 1 & \zeta \end{array} \right|^2 , \tag{3.19}$$

However, provided the integrand remains finite as t, x, y and z approach values for which degenerate intermediate channels are allowed, the integral as a whole is continuous (and remains crossing symmetric), and so continuity of the whole correlation function implies that the couplings to the degenerate blocks $\tilde c(t,x,p)$ are all zero. In particular the 'vacuum' channel $p = 0$ never occurs as an intermediate channel.

We note here that we have not proven that the integrand remains finite, and indeed there are limiting values of t, x, y, z for which the limits of the chiral blocks develop poles, but in all cases we have investigated these poles are cancelled by zeros in the structure constants.

3.3 Two point function on the upper half plane

Rather than working with the unit disc, it is convenient to use a conformal mapping to the upper half plane. On the UHP we can use directly a method introduced by Cardy [5] to express the two point function in terms of chiral four point blocks (3.5).

Since in section 2 we have only presented the one point function (2.13) in the presence of a boundary, we can only say how a bulk field couples to the identity on the boundary. To describe the coupling to other boundary fields we would need correlators involving one bulk and one boundary field.

In order to check crossing symmetry in presence of a boundary we have to restrict ourselves for the time being to boundary conditions on which the unique primary boundary field is the identity. Among the set of boundary conditions presented in section 2 there is exactly one fulfilling the criterion, the boundary condition $\hat{1}$.

The statement of crossing symmetry for the two point function on the UHP is depicted in figure 2*b*. On the one hand we can expand both bulk fields in terms of boundary fields, i.e. only the identity in our case. On the other hand we can compute the OPE in the bulk and evaluate the one point function of the resulting field.

Denote as ${}^{\hat a}B_x{}^1$ the constant describing the coupling of a bulk field to the identity on the $\hat a$-boundary, i.e.

$$\frac{\langle \phi_x(\zeta,\bar\zeta)\rangle^{\hat a}_{\rm disc}}{Z^{\hat a}} = {}^{\hat a}B_x{}^1 \cdot (1-|\zeta|^2)^{-2h_x}\,, \qquad {}^{\hat a}B_x{}^1 = (-1)^{[x]} \cdot \frac{\sin(\pi a x)}{\pi a}\,, \tag{3.20}$$

where the numerical value of ${}^{\hat a}B_x{}^1$ follows from (2.13) and (2.14). Since unit disc and UHP are related by a conformal transformation one sees that the values of the bulk boundary couplings are the same in both cases.

Coupling both bulk fields to the identity on the boundary we find the correlator

$$
\begin{aligned}
A(a,b) &= \frac{\langle \phi_x(a{+}ib)\,\phi_y(ib)\,\rangle^{\hat{1}}_{\mathrm{UHP}}}{Z} \\
&= (2b)^{-2h_x-2h_y}\,\eta^{2h_y}\cdot\, {}^{\hat{1}}B_x{}^{1}\;{}^{\hat{1}}B_y{}^{1}\cdot\, \begin{array}{c|c|c} & x & y \\ x & \mathbb{1} & y \\ \hline & 1 & \eta \end{array}\,, \qquad \eta = \frac{4b^2}{a^2+4b^2}\,, \qquad (3.21)
\end{aligned}
$$

Taking the OPE of the two bulk fields first we obtain

$$
\begin{aligned}
B(a,b) &= \frac{\langle\, \phi_x(a{+}ib)\,\phi_y(ib)\,\rangle^{\hat{1}}_{\mathrm{UHP}}}{Z} \\
&= (2b)^{-2h_x-2h_y}\,\eta^{2h_y}\cdot \int_{\mathbb{S}} \mathrm{d}p\;\; c(x,y,p)\;{}^{\hat{1}}B_p{}^{1}\cdot\, \begin{array}{c|c|c} & y & y \\ x & p & x \\ \hline & 1 & 1{-}\eta \end{array}\,, \qquad (3.22)
\end{aligned}
$$

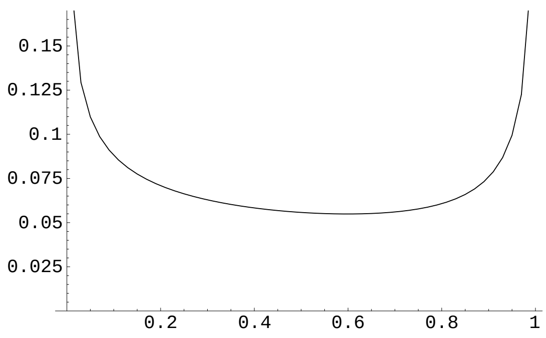
where η is the same as in equation (3.21). Note that the conformal block in (3.21) involves a degenerate representation in the intermediate channel, while this is not the case for the blocks appearing in (3.22).

One can now verify crossing symmetry by checking that $A(a,b) = B(a,b)$ as a function of the two real parameters a, b. Instead of looking at A and B, we can consider the functions

$$
A_b(\eta) = (2b)^{2h_x+2h_y}\eta^{-2h_y}A(a,b) \quad \text{and} \quad B_b(\eta) = (2b)^{2h_x+2h_y}\eta^{-2h_y}B(a,b)\,, \quad (3.23)
$$

which only depend on the crossing ratio η.

We again find excellent numerical agreement between $A_b(\eta)$ and $B_b(\eta)$, an example is given in figure 4.



ζ	$A_b(\zeta)$	$\lvert A_b(\zeta){-}B_b(\zeta)\rvert$
0.04	0.11798728 ...	0.00003262 ...
0.11	0.08548602 ...	0.00000313 ...
0.18	0.07325737 ...	0.00000054 ...
0.32	0.06177076 ...	0.00000000 ...
0.53	0.05532286 ...	0.00000022 ...
0.67	0.05543060 ...	0.00000069 ...
0.81	0.06154874 ...	0.00002715 ...
0.88	0.07096936 ...	0.00017047 ...
0.95	0.09943759 ...	0.00198972 ...

Figure 4: Example of crossing on the upper half plane with $x = 1.25$ and $y = 0.8$. The data has been obtained by using the c-recursion relation (E.10) for internal channel zero to level 8. In the crossed channel the H-recursion (E.11) was used to level 6 together with a numerical integration step width of 0.005. The plot shows the function $A_b(\zeta)$ as given in (3.23) against ζ.

4. The limit of minimal models

This more technical section treats in some detail how we define the $c = 1$ theory

$$M_\infty = \lim_{p\to\infty} M_p\,, \tag{4.1}$$

In particular it is shown how the expressions (2.3) and (2.4) can be obtained. The contents of this section should be understood as a motivation for (2.3), (2.4) and does not constitute a proof.

As a first step it is shown that the minimal model bulk spectrum approaches a continuum as $c \to 1$. Then the M_∞ fields are defined as an average over minimal model bulk fields of approximately the same weight. This prescription is used to compute the correlators of M_∞.

4.1 Limit of the spectrum

Consider the unitary minimal model M_p. With $t = \frac{p}{p+1}$ and $\Delta = 1-t = \frac{1}{p+1}$ its central charge is given by

$$c = 13 - 6(t + t^{-1}) = 1 - 6\Delta^2 + O(\Delta^3) < 1 \tag{4.2}$$

The limit $c \to 1$ corresponds to $\Delta \to 0$ or equivalently to $p \to \infty$.

The bulk field content of the diagonal modular invariant is given by the Kac-table: There is one bulk field $\phi_{r,s}(\zeta,\bar\zeta)$ for each pair of integers r, s with $1 \le r \le p-1$ and $1 \le s \le p$, subject to the identification $(r, s) \sim (p-r, p+1-s)$. With $d_{ab} = a - bt$ the left/right conformal weight of $\phi_{r,s}$ is given by

$$h_{r,s} = h_{p-r,p+1-s} = \frac{1}{4t}\big({d_{rs}}^2 - {d_{11}}^2\big)\,, \tag{4.3}$$

Taking $\Delta \to 0$, each individual weight approaches the limiting value

$$h_{r,s} \xrightarrow{\Delta\to 0} \frac{(r-s)^2}{4}\,, \tag{4.4}$$

However, while each weight approaches one of these discrete values, they do so in such a way that the spectrum as a whole approaches a continuum.

To see this, we first pick a range K of Kac-labels such that each field is represented once. It will be convenient to choose the subset above the diagonal, that is we choose the set

$$K = \big\{(r,s)\,\big|\, 1 \le r \le p-1\,,\ 1 \le s \le p\,,\ d_{rs}{>}0\big\}\,, \tag{4.5}$$

Recall that the M_∞ fields $\phi_x(\zeta,\bar\zeta)$ were labelled by a real number x, which was linked to the conformal weight via $h_x = x^2/4$. Let $x_{rs} = 2\sqrt{h_{rs}}$. The set K can be split into slices which contain fields with approximately the same weight. We will parametrise these slices by the M_∞ label x: denote as $N(x,\varepsilon)$ the subset of K which contains all labels (r,s) with x_{rs} in the interval $[x,x{+}\varepsilon[$,

$$N(x,\varepsilon) = \big\{(r,s) \in K \bigm| x \le x_{rs} < x+\varepsilon\big\}\,, \tag{4.6}$$

We will need a more direct way to describe the Kac labels $(r,s) \in N(x,\varepsilon)$. Working to first order in Δ, we find

$$x_{rs} = r{-}s + \Delta\frac{r+s}{2} + O(\Delta^2)\,, \tag{4.7}$$

From the possible ranges of r,s, i.e. $1 \le r \le p{-}1$ and $1 \le s \le p$, it follows that $0 < \Delta\frac{r+s}{2} < 1$. Denoting again by $[x]$ the largest integer less or equal to x, we can thus write

$$x_{rs} = [x_{rs}] + f_{rs} \quad \text{where} \quad [x_{rs}] = r-s\,, \quad f_{rs} = \Delta\cdot s + \Delta\tfrac{[x_{rs}]}{2}\,, \tag{4.8}$$

Looking again at the allowed ranges of r,s we find that the range of f_{rs} is restricted to

$$\Delta\cdot\frac{2+[x_{rs}]}{2} \le f_{rs} \le 1-\Delta\cdot\frac{4+[x_{rs}]}{2}\,, \tag{4.9}$$

For fixed Δ there is thus an interval around each integer which is not accessible in the given minimal model. Outside this interval (4.8) shows that the values of x_{rs} are distributed uniformly (up to $O(\Delta^2)$) with separation Δ.

The above shows that for $x \notin \mathbb{Z}$ and for Δ,ε small enough, the following more direct description of the set $N(x,\varepsilon)$ is possible:

$$N(x,\varepsilon) = \left\{([x]+n,n) \in K \bigm| n \in \mathbb{Z} \text{ and } \frac{f_x}{\Delta} + \frac{[x]}{2} \le n < \frac{f_x+\varepsilon}{\Delta} + \frac{[x]}{2}\right\}, \tag{4.10}$$

Here f_x denotes again the fractional part of x, i.e. $f_x = x-[x]$. In other words, the set $N(x,\varepsilon)$ thus simply consists of Kac labels (r,s) where s takes all values in a certain range and the difference $r-s$ is fixed to be $[x]$.

In particular (4.10) shows that the number of elements in each set $N(x,\varepsilon)$ is given by

$$|N(x,\varepsilon)| \sim \frac{\varepsilon}{\Delta}\,, \tag{4.11}$$

For Δ small enough, we can find arbitrarily many labels in the set $N(x,\varepsilon)$, that is arbitrarily many bulk fields $\phi_{r,s}$ have a conformal weight $h_{r,s}$ close to h_x. Furthermore the leading behaviour of $|N(x,\varepsilon)|$ is independent of x. Thus the spectrum of primary fields approaches a continuum with a uniform spectral density, when parametrised by x. Had we chosen the conformal weight h as the parameter instead of x, the spectral density would not be constant and less convenient to work with.

4.2 Normalisation of M_p

In order to get finite expressions for the quantities of our interest in the limit $\Delta \to 0$, we have to normalise the individual unitary minimal models M_p in an appropriate way.

To do so we first choose a particular solution for the minimal model structure constants. Then free parameters are inserted, which reflect the freedom to redefine the fields.

As already mentioned in section 4.1, the minimal model M_p has one bulk field $\phi_i(\zeta, \bar{\zeta})$ for each pair of Kac labels $i \in K$, where K is defined as in (4.5). For these we need to specify the bulk structure constants ${C_{ij}}^k$ that appear in the OPE of two primary fields

$$\phi_i(\zeta,\bar{\zeta})\phi_j(0,0) = \sum_k {C_{ij}}^k \, |\zeta|^{2h_k-2h_i-2h_j} (\phi_k(0,0) + \text{descendant fields}) \,, \qquad (4.12)$$

We are also interested in correlation functions in the unit disc. To compute these we first need to know the set of conformal boundary conditions for the model M_p.

We will distinguish between fundamental boundary conditions and superpositions. Fundamental boundary conditions can not be expressed as a sum (with positive integer coefficients) of other conformal boundary conditions. All superpositions on the other hand are sums of fundamental boundary conditions. In terms of boundary fields, fundamental boundary conditions are identified by the property that exactly one field of conformal weight zero lives on them.

The fundamental conformal boundary conditions of M_p have been found by Cardy [6]. There is a finite number of them, and they are in one to one correspondence with the Kac labels in K. So we can label boundary conditions by $a \in K$.

The correlators on the unit disc with boundary condition a are now specified once we know the (unnormalised) one point functions of the primary bulk fields $\langle \phi_i(0,0) \rangle^a_{\text{disc}}$ and the unit disc partition function Z^a.

The solution for M_p that serves as a starting point uses the bulk structure constants ${C_{ij}}^k$ by Dotsenko and Fateev, discussed below in section 4.5 and the unit disc expectation values given by Cardy and Lewellen [7]. In our normalisation these have the properties

$$\langle 0|0 \rangle = 1 \,, \quad {C_{ii}}^1 = 1 \,, \quad Z^a = \frac{{S_1}^a}{\sqrt{{S_1}^1}} \,, \quad \langle \phi_i(0,0) \rangle^a_{\text{disc}} = \sqrt{\frac{{S_1}^1}{{S_i}^1}} \cdot \frac{{S_i}^a}{{S_1}^a} \cdot Z^a \,, \qquad (4.13)$$

Here S_i^j denotes the modular S-matrix that appears in the transformation of characters. For minimal models it is given by

$$ {S_{rs}}^{r's'} = 2^{3/2} \big(p(p{+}1)\big)^{-1/2} (-1)^{1+rs'+sr'} \cdot \sin(\pi r r'/t) \cdot \sin(\pi s s' t) \,, \qquad (4.14)$$

The solution (4.13) is in fact unique, up to the freedom to rescale the fields and a choice of normalisation for the bulk vacuum,

$$\phi_i(\zeta,\bar\zeta) \to \alpha_i \cdot \phi_i(\zeta,\bar\zeta)\,, \qquad |0\rangle \to \gamma \cdot |0\rangle\,, \tag{4.15}$$

Implementing this choice of normalisation explicitly we can rewrite (4.13) as

$$\begin{aligned} &\langle 0|0\rangle = \gamma(\Delta)^2\,, \quad C_{ii}{}^1 = \alpha_i(\Delta)^2\,, \quad C_{ij}{}^k = \frac{\alpha_i(\Delta)\alpha_j(\Delta)}{\alpha_k(\Delta)} \cdot \tilde C_{ij}^{\,k}\,, \\ &Z^a = \gamma(\Delta) \cdot \frac{S_1{}^a}{\sqrt{S_1{}^1}}\,, \quad \langle \phi_i(0,0)\rangle^a_{\text{disc}} = \alpha_i(\Delta) \cdot \sqrt{\frac{S_1{}^1}{S_i{}^1}} \cdot \frac{S_i{}^a}{S_1{}^a} \cdot Z^a\,, \end{aligned} \tag{4.16}$$

Here $\tilde C_{ij}^{\,k}$ refers to the bulk structure constants in the normalisation (4.13), i.e. $\tilde C_{ii}^{\,1}$=1.

One may wonder how the normalisation of the bulk vacuum enters the unit disc amplitude Z^a. This is due to the boundary state formalism. If we describe the boundary of the unit disc as an out state $\langle a|$, then we would like it to reproduce the unit disc expectation values given above

$$\langle \phi_i(0,0)\rangle^a_{\text{disc}} = \langle a|i\rangle\,, \qquad Z^a = \langle a|0\rangle\,, \tag{4.17}$$

Furthermore, the normalisation of the boundary state $\langle a|$ is constrained by the cylinder partition function. Consider a cylinder of length R and circumference L with conformal boundary conditions a,b. Its partition function in the open and closed string channel is given by, respectively, [6]

$$Z_{\text{cyl}} = \langle a| e^{\frac{-2\pi R}{L}(L_0+\bar L_0-\frac{c}{12})}|b\rangle = \sum_i N_{ia}{}^b \chi_i(e^{-\pi\, L/R})\,, \tag{4.18}$$

Here χ_i denotes the character of the Virasoro highest weight representation of weight h_i and $N_{ij}{}^k$ are the Verlinde fusion numbers. Note that the r.h.s. of this expression is independent of any normalisation. If we take the large R limit on the lhs, only the ground state will contribute and we get

$$Z_{\text{cyl}} \sim \frac{\langle a|0\rangle\langle 0|b\rangle}{\langle 0|0\rangle} \cdot e^{-RE_0(L)}\,, \tag{4.19}$$

where $E_0(L) = -c/6L$. Requiring (4.19) to be independent of the choice of γ in (4.15), together with (4.17), leads to the γ-dependence as in (4.16).

The parameters $\alpha_i(\Delta)$ and $\gamma(\Delta)$ introduced in the solution (4.16) will be fixed in the next section to obtain the desired behaviour of correlators in the $\Delta \to 0$ limit.

4.3 Correlators of averaged fields

The basic idea in taking the $\Delta \to 0$ limit in the bulk is that the fundamental parameters of minimal models are not the Kac labels (r,s), but the conformal weights

h_{rs}. So rather than taking the limit for a fixed pair of Kac labels (r, s) we take it for a fixed value of h. This means that the pairs (r, s) now depend on Δ in such a way that $h_{rs} \to h$ as $\Delta \to 0$.

In fact, choosing a specific sequence of Kac labels to approach a given weight h may cause problems. Consider for example the sequences $i_p = \big(2[\frac{p}{5}], 2[\frac{p}{5}]\big)$ and $j_p = \big(2[\frac{p}{5}]+1, 2[\frac{p}{5}]+1\big)$. Here the notation $[x]$ refers again to the largest integer $\leq x$. Both, i_p and j_p are Kac labels in the minimal model M_p. Both sequences approach the limiting weight $h = 1/25$. But the fusion $i_p \times i_p \to i_p$ is forbidden, while the fusion $j_p \times j_p \to j_p$ is allowed. So taking the limit of the bulk structure constants $C_{ii}{}^{i}$ will automatically give zero, while the limit of $C_{jj}{}^{j}$ is potentially nonzero.

To avoid this sort of difficulty one can consider averages over primary fields. That is, instead of taking limits of individual fields, in each model M_p we sum over all fields whose weight lies in a certain neighbourhood of h.

For the computation below it is preferable to use the parameter $x = 2\sqrt{h}$ instead of h itself, since as we have seen in section 4.1 the spectrum then approaches the continuum with a uniform density. The averaged fields $V_{x,\varepsilon}(\zeta, \bar{\zeta})$ we will consider are defined as sums over primary fields with Kac labels in the set $N(x, \varepsilon)$:

$$V_{x,\varepsilon}(\zeta, \bar{\zeta}) = A_x(\Delta, \varepsilon) \cdot \sum_{rs \in N(x,\varepsilon)} \phi_{r,s}(\zeta, \bar{\zeta}) \,. \tag{4.20}$$

The prefactor $A_x(\Delta, \varepsilon)$, together with $\alpha_i(\Delta)$ and $\gamma(\Delta)$ from section 4.2 will be adjusted later to get finite answers for the correlators. It will turn out that A_x can be chosen to be independent of x and α_i independent of i. This means we can restrict the set of possible gauge choices to $A_x(\Delta, \varepsilon) = A(\Delta, \varepsilon)$ and $\alpha_i(\Delta) = \alpha(\Delta)$, for some A, α, and still obtain finite answers for the n-point functions.

To obtain the M_∞ fields $\phi_x(\zeta, \bar{\zeta})$ defined in section 2, one starts from the averaged fields $V_{x,\varepsilon}(\zeta, \bar{\zeta})$ and first takes the limit $\Delta \to 0$, and then $\varepsilon \to 0$

$$\phi_x(\zeta, \bar{\zeta}) = \lim_{\varepsilon \to 0} \lim_{\Delta \to 0} V_{x,\varepsilon}(\zeta, \bar{\zeta}) \,, \tag{4.21}$$

Note that it does not make sense to exchange the limits, since at finite Δ only finitely many primary fields are present in the model, and taking ε to zero would in general leave no fields in the interval $[x, x{+}\varepsilon]$ to be averaged over.

For finite Δ we would expect the averaged fields $V_{x,\varepsilon}$ to be in some sense 'close' to the analogous average over fields at $\Delta = 0$,

$$V_{x,\varepsilon}(\zeta, \bar{\zeta}) \approx \frac{1}{\varepsilon} \int_x^{x+\varepsilon} \mathrm{d}x' \; \phi_{x'}(\zeta, \bar{\zeta}) \,, \tag{4.22}$$

This is the prescription we will use to compute the various correlators below.

4.3.1 Bulk two point function

For the bulk two point function in M_∞ we would like the property

$$\langle \phi_x(\zeta,\bar{\zeta})\, \phi_y(0,0) \rangle = \delta(x-y) \cdot |\zeta|^{-4h_x}\,, \tag{4.23}$$

From here on we suppress the coordinate dependence of the correlators for clarity. The letters x, y, z always label fields, and never coordinates. In terms of (4.22) the desired behaviour for the two point function implies that

$$\frac{1}{\varepsilon^2}\int_x^{x+\varepsilon} \mathrm{d}x' \int_y^{y+\varepsilon} \mathrm{d}y'\, \langle \phi_{x'}\, \phi_{y'} \rangle = \frac{\varepsilon - |x-y|}{\varepsilon^2} \cdot \theta(\varepsilon - |x-y|)\,, \tag{4.24}$$

Inserting the definition (4.20) we find that for minimal models

$$\begin{aligned} \langle\, V_{x,\varepsilon}\, V_{y,\varepsilon}\,\rangle \; &= \sum_{i \in N(x,\varepsilon)} \sum_{j \in N(y,\varepsilon)} A(\Delta,\varepsilon)^2 \cdot \langle\, \phi_i\, \phi_j\,\rangle \; = \sum_{i \in N(x,\varepsilon) \cap N(y,\varepsilon)} A^2 \cdot \alpha(\Delta)^2 \cdot \gamma(\Delta)^2 \\ &\approx A^2 \cdot \alpha^2 \cdot \gamma^2 \cdot \frac{\varepsilon - |x-y|}{\Delta} \cdot \theta(\varepsilon - |x-y|)\,, \end{aligned} \tag{4.25}$$

To obtain the desired behaviour (4.23) we thus need

$$A^2 \cdot \alpha^2 \cdot \gamma^2 = \frac{\Delta}{\varepsilon^2}\,, \tag{4.26}$$

4.3.2 Unit disc one point function

As already mentioned in section 4.2 the set of distinct fundamental conformal boundary conditions is given by K as defined in (4.5). Using the results of [22] it was argued in [16] that in the limit $c \to 1$ not all boundary conditions remain independent. Instead one has the formal relations

$$\lim_{c\to 1}\,(r,s) = \lim_{c\to 1} \sum_{a \in r\otimes s} (a,1)\,, \tag{4.27}$$

Here $a{\in}r{\otimes}s$ is the short hand notation introduced in equation (2.12).

In words the meaning of (4.27) is that in the $c \to 1$ limit a general boundary condition (r,s) becomes indistinguishable form a superposition of boundary conditions of the type $(a,1)$ and is thus no longer a fundamental boundary condition itself.

To find all boundary conditions in M_∞ that appear in this limit of minimal models, it is thus enough to consider the $(a,1)$-boundary conditions. We will denote the $c \to 1$ limit of $(a,1)$ as $\hat{a}$.

To obtain the limit of the boundary one point functions we can directly use the definition (4.21)

$$\langle \phi_x(\zeta,\bar{\zeta}) \rangle^{\hat{a}}_{\text{disc}} = \lim_{\varepsilon\to 0}\lim_{\Delta\to 0}\, \langle V_{x,\varepsilon}(\zeta,\bar{\zeta}) \rangle^{(a,1)}_{\text{disc}} = \lim_{\varepsilon,\Delta} \sum_{i \in N(x,\varepsilon)} A \cdot \langle \phi_i(\zeta,\bar{\zeta}) \rangle^{(a,1)}_{\text{disc}}\,, \tag{4.28}$$

In order to compute the average using the expressions in (4.16) we need to consider two different limits of the S-matrix (4.14). In the first limit we keep the Kac labels fixed and take Δ to zero. The leading behaviour of ${S_{11}}^{a1}$ in this case can be found directly from (4.14) to be

$$ {S_{11}}^{a1} \sim 2^{3/2}\pi^2\Delta^3 \cdot a\,, \tag{4.29} $$

In the second limit we consider the leading behaviour as Δ goes to zero of ${S_{rs}}^{a1}$ where r,s varies with Δ such that $x_{rs} = x$ is kept (approximately) fixed. To compute this limit recall from (4.7) that $x_{rs} = r - st + \Delta \cdot [x]/2$ and note that (4.14) can be rewritten in the form

$$ {S_{rs}}^{jk} = 2^{3/2}\big(p(p+1)\big)^{-1/2} \cdot \sin(\pi j(r-st)/t) \cdot \sin(\pi k(r-st))\,, \tag{4.30} $$

The leading behaviour of ${S_{rs}}^{a1}$ with $x_{rs} = x$ kept fixed is given by

$$ {S_{rs}}^{a1} \sim 2^{3/2}\Delta \cdot \sin(\pi x)\sin(\pi a x)\,, \tag{4.31} $$

Using (4.29) and (4.31) we can compute the leading behaviour of the unit disc partition function (4.13) and the averaged one point function (4.28) to be

$$ \begin{aligned} Z^{\hat{a}} &\sim 2^{3/4}\,\pi\,a\cdot\gamma\cdot\Delta^{3/2}\,,\\ \langle\phi_x\rangle^{\hat{a}}_{\rm disc} &\sim 2^{3/4}\,(-1)^{[x]}\,\sin(\pi a x)\cdot A\,\alpha\,\gamma\cdot\Delta^{-1/2}\,\varepsilon\,, \end{aligned} \tag{4.32} $$

If we want $\langle\phi_x(0,0)\rangle^{\hat{a}}_{\rm disc}$ to remain finite in the limit we have to require

$$ A\,\alpha\,\gamma \sim ({\rm const})\cdot\frac{\Delta^{1/2}}{\varepsilon}\,, \tag{4.33} $$

4.3.3 Bulk three point functions

To find the bulk correlator of three M_∞ fields we use again the prescription (4.21). For clarity we drop the coordinate dependence in the following formulae for the OPEs, the parameters x,y,z refer to M_∞ field labels:

$$ \langle\,\phi_x\,\phi_y\,\phi_z\rangle = \lim_{\varepsilon\to0}\lim_{\Delta\to0}\sum_{i,j,k} A^3\,{C_{jk}}^i\;{C_{ii}}^1\,\langle 0|0\rangle = \lim_{\varepsilon,\Delta} A^3\alpha^3\gamma^2\sum_{i,j,k}\tilde{C}_{ij}^{\,k}\,, \tag{4.34} $$

where the sum runs over $i\in N(x,\varepsilon)$, $j\in N(y,\varepsilon)$ and $k\in N(z,\varepsilon)$.

To compute the leading behaviour of the sum (4.34), the most important input is an observation due to Dotsenko [9]: The normalised structure constants $\tilde{C}_{ij}^{\,k}$ are continuous in the parameters $d_i = r_i - s_i t$, $d_j = r_j - s_j t$, $d_k = r_k - s_k t$. That is, if the index combination is allowed by the fusion rules, structure constants with similar d_i, d_j, d_k will have a similar value. Furthermore they do stay finite in the limit $\Delta\to0$.

So, as was the case for the one point functions on the unit disc, to work out the sum in (4.34), we can treat $\tilde{C}_{ij}^{\,k}$ as independent of i,j,k, provided that we can estimate how many triples in the set $N(x,\varepsilon)\times N(y,\varepsilon)\times N(z,\varepsilon)$ are allowed by fusion. In section 4.4 we will argue that in the limit $\varepsilon,\Delta\to 0$, depending on the choice of x,y,z either no triples are allowed or exactly half of them.

This gives the leading behaviour

$$\langle\, \phi_x\, \phi_y\, \phi_z\,\rangle \sim A^3\alpha^3\gamma^2\cdot\frac{\varepsilon^3}{\Delta^3}\cdot\frac{\tilde{C}_{ij}^{\,k}}{2}\,, \tag{4.35}$$

where the triple (i,j,k) is taken from the set $N(x,\varepsilon)\times N(y,\varepsilon)\times N(z,\varepsilon)$. If this set contains any triples allowed by the fusion rules, we assume that (i,j,k) is chosen to be one of them.

Demanding the bulk three point functions to have a finite limit leads to

$$A^3\alpha^3\gamma^2 = (\text{const})\cdot\frac{\Delta^3}{\varepsilon^3}\,, \tag{4.36}$$

4.3.4 Collection of results

In order for the unit disc one point function and the bulk two and three point functions to have a finite limit as $\Delta\to 0$, we found the conditions (4.26), (4.33) and (4.36):

$$A^2\,\alpha^2\,\gamma^2 = \frac{\Delta}{\varepsilon^2}\,,\quad A\,\alpha\,\gamma = G\cdot\frac{\Delta^{1/2}}{\varepsilon}\,,\quad A^3\,\alpha^3\,\gamma^2 = H\cdot\frac{\Delta^3}{\varepsilon^3}\,, \tag{4.37}$$

for some constants G,H. These conditions have the unique solution

$$G=1\,,\qquad A\,\alpha = H\cdot\frac{\Delta^2}{\varepsilon}\,,\qquad \gamma=\frac{1}{H}\cdot\Delta^{-3/2}\,, \tag{4.38}$$

Since only the combination $A\cdot\phi_i$ enters in the calculation of the limiting correlators, it is not surprising that the limit depends on the combination $A\cdot\alpha$. For convenience we choose $\alpha=1$ and $A=H\cdot\Delta^2/\varepsilon$. Different choices of H correspond to different normalisations of M_∞. To reproduce the formulae in section 2, one needs to choose $H=1$.

The $\Delta\to 0$ limit then yields the following expressions:

$$\begin{aligned}
&\text{unit disc partition function :} && Z^{\hat a} = 2^{3/4}\pi a\\
&\text{unit disc one point function :} && \langle\phi_x(\zeta,\bar\zeta)\rangle^{\hat a}_{\rm disc} = 2^{3/4}\,(-1)^{[x]}\,\sin(\pi a x)\,(1-|\zeta^2|)^{-x^2/2}\\
&\text{bulk two point function :} && \langle\phi_x(\zeta,\bar\zeta)\,\phi_y(0,0)\rangle = \delta(x-y)\cdot|\zeta|^{-x^2}\,,\\
&\text{bulk three point function :} && \langle\phi_x(\zeta_1,\bar\zeta_1)\,\phi_y(\zeta_2,\bar\zeta_2)\,\phi_z(\zeta_3,\bar\zeta_3)\rangle = \frac{1}{2}\,\tilde{C}_{ij}^{\,k}\,\times\\
& && \times\,|\zeta_{12}|^{(z^2-x^2-y^2)/2}\,|\zeta_{13}|^{(y^2-x^2-z^2)/2}\,|\zeta_{23}|^{(x^2-y^2-z^2)/2}
\end{aligned} \tag{4.39}$$

where $\tilde{C}_{ij}^{\,k}$ is understood as in (4.35).

As a consistency check we can verify that the decomposition of boundary states (4.27) holds for the one point functions. If we repeat the calculation in (4.28) for the general boundary condition (b,c) instead of $(a,1)$ we find

$$\langle\phi_x(\zeta,\bar\zeta)\rangle^{(b,c)}_{\rm disc} = 2^{3/4}\,(-1)^{[x]}\,\frac{\sin(\pi bx)\sin(\pi cx)}{\sin(\pi x)}\,(1-|\zeta^2|)^{-x^2/2} = \sum_{a\in b\otimes c}\langle\phi_x(\zeta,\bar\zeta)\rangle^{\hat a}_{\rm disc}\,, \tag{4.40}$$

in accordance with (4.27).

In the following two sections the factor of a half in (4.35) is justified and it is shown how analytic expressions can be obtained for the $\Delta\to 0$ limit of $\tilde C_{ij}^{\,k}$.

4.4 Fusion rules

In the following we want to estimate how many triples of Kac-labels in the product $N(x,\varepsilon)\times N(y,\varepsilon)\times N(z,\varepsilon)$ are allowed by fusion. It is argued that, depending on x,y,z and for ε sufficiently small, either no triples are allowed, or exactly half of them.

The minimal model fusion rules state that $(r,s)\times(u,v)$ can fuse to (a,b) if and only if

$$\mathcal{N}_{ru}^{a}(p)\cdot\mathcal{N}_{sv}^{b}(p+1)+\mathcal{N}_{ru}^{p-a}(p)\cdot\mathcal{N}_{sv}^{p+1-b}(p+1)=1\,, \tag{4.41}$$

where

$$\mathcal{N}_{ab}^{c}(r)=\begin{cases}1: & |a-b|<c<\min(a+b,2r-a-b)\,,\ a+b+c \text{ odd}\\ 0: & \text{otherwise}\end{cases}$$

In order to see what is happening in the $\varepsilon,\Delta\to 0$ limit it is helpful to reformulate these rules somewhat.

Recall the parametrisation of the set $N(x,\varepsilon)$ as given in (4.10). There it was found that, for a given $x=[x]+f_x$,

$$x_{rs}\in[x,x{+}\varepsilon]\quad\Leftrightarrow\quad r=[x]{+}s \text{ and } s\cdot\Delta=f_x{+}\Delta\tfrac{[x]}{2}{+}\nu_x \text{ where } \nu_x\in[0,\varepsilon]\,, \tag{4.42}$$

Using these relations, we can eliminate r,u,a from equation (4.41), yielding expressions in x,y,z and s,v,b. Then we multiply the inequalities by Δ and neglect terms of order Δ and ε. Consider the first term in (4.41). Applying this procedure, $\mathcal{N}_{ru}^{a}(p)=1$ is equivalent to saying, up to terms of order Δ and ε,

$$|f_x{-}f_y|<f_z<\min(f_x{+}f_y\,,\,2{-}f_x{-}f_y) \text{ and } [x]+[y]+[z]+s+v+b \text{ odd}\ , \tag{4.43}$$

Similarly, $\mathcal{N}_{sv}^{b}(p{+}1)$ is nonzero if and only if

$$|f_x{-}f_y|<f_z<\min(f_x{+}f_y\,,\,2{-}f_x{-}f_y) \text{ and } s+v+b \text{ odd}\ , \tag{4.44}$$

It follows that the first summand in (4.41) can be nonzero only if $[x]+[y]+[z]$ is even. If this is the case, the indices s, v, b have to be adjusted, s.t. $s+v+b$ is odd. If $|N(x,\varepsilon)| \sim \varepsilon/\Delta$ is sufficiently large, i.e. Δ sufficiently small, then this condition will in good approximation be true for half the triples in $N(x,\varepsilon) \times N(y,\varepsilon) \times N(z,\varepsilon)$.

For the second summand in (4.41) we find in the same way, that it is nonzero only if

$$|f_x - f_y| < 1 - f_z < \min(f_x + f_y\,,\, 2 - f_x - f_y) \text{ and } [x]+[y]+[z] \text{ odd}, \qquad s+v+b+p \text{ even}\,, \tag{4.45}$$

The two conditions (4.45) and (4.44) give rise to the step function $P(x,y,z)$ given in (2.5).

The statement for the fusion rules is that, if $x, y \notin \mathbb{Z}$, then we can choose ε and Δ small enough, s.t. $P(x,y,z) = 0$ implies that fields of $N(x,\varepsilon) \times N(y,\varepsilon)$ cannot fuse to fields in $N(z,\varepsilon)$. Conversely, if $P(x,y,z) = 1/2$, in the limit $\Delta \to 0$, $\varepsilon \to 0$ while letting ε/Δ go to infinity, the number of allowed triples will tend to a half.

4.5 Bulk structure constants

In this section we take the $\Delta \to 0$ limit of the structure constants $\tilde{C}_{ij}^{\,k}$, where we assume that ${N_{ij}}^k = 1$, i.e. the triple is allowed by the fusion rules.

Define, as before in the minimal model M_p the abbreviations $d_{rs} = r - st$, where[3] $t = p/(p+1)$. The minimal model structure constants found by Dotsenko and Fateev in [10, 11, 12] are

$$\begin{aligned} C_{(rs)(uv)}^{\;(ab)} =& t^{4(k-1)(\ell-1)} \cdot \prod_{m=1}^{k-1}\prod_{n=1}^{\ell-1} \Big\{ d_{mn}(d_{mn}-d_{rs})(d_{mn}-d_{uv})(d_{mn}+d_{k\ell}-d_{11}-d_{uv}-d_{rs}) \Big\}^{-2} \times \\ & \times \prod_{m=1}^{k-1} \frac{\Gamma(m/t)\,\Gamma\big((m-d_{rs})/t\big)\,\Gamma\big((m-d_{uv})/t\big)\,\Gamma\big((d_{11}+d_{rs}+d_{uv}-d_{k\ell}-d_{m\ell})/t\big)}{\Gamma(1-m/t)\,\Gamma\big((d_{rs}-d_{m1})/t\big)\,\Gamma\big((d_{uv}-d_{m1})/t\big)\,\Gamma\big((d_{k\ell}+d_{m\ell}-d_{rs}-d_{uv}-d_{11}+t)/t\big)} \times \\ & \times \prod_{n=1}^{\ell-1} \frac{\Gamma(nt)\,\Gamma(d_{rs}+nt)\,\Gamma(d_{uv}+nt)\,\Gamma\big(d_{k\ell}+d_{kn}-d_{rs}-d_{uv}-d_{11}\big)}{\Gamma(1-nt)\,\Gamma(d_{1n}-d_{rs})\,\Gamma(d_{1n}-d_{uv})\,\Gamma\big(d_{rs}+d_{uv}+d_{11}-d_{k\ell}-d_{kn}+1\big)}\,, \end{aligned} \tag{4.46}$$

where $k = (r+u-a+1)/2$, $\ell = (s+v-b+1)/2$. The indices have to satisfy the following fusion constraints:

$$\begin{aligned} &|r-u|+1 \le a \le \min(r+u-1, 2p-r-u-1) \qquad \text{and } r+u-a+1 \in 2\mathbb{Z} \\ &|s-v|+1 \le b \le \min(s+v-1, 2q-s-v-1) \qquad \text{and } s+v-b+1 \in 2\mathbb{Z}\,, \end{aligned} \tag{4.47}$$

As given in (4.46), the structure constants do not obey the normalisation condition (4.13), i.e. in general $C_{(rs)(rs)}^{\;(11)} \neq 1$. To change this one could rescale the fields in

[3]The formula (4.46) for the bulk structure constants also holds for non-unitary minimal models $M(p,q)$. In this case one simply takes $t = p/q$.

such a way that

$$C_{(rs)(uv)}^{\quad(ab)} \to \frac{\sqrt{C_{(ab)(ab)}^{\quad(11)}}}{\sqrt{C_{(rs)(rs)}^{\quad(11)}\, C_{(uv)(uv)}^{\quad(11)}}} \cdot C_{(rs)(uv)}^{\quad(ab)}\,, \tag{4.48}$$

This is done explicitly in [12]. In the present case this will however not be necessary, since for $c \to 1$, the prefactor turns out to approach one: $\lim_{t\to 1} C_{(rs)(rs)}^{\quad(11)} = 1$. In the following we will thus continue to work with the original expression (4.46).

The limiting procedure relies on two observations by Dotsenko [9].

Firstly, the numbers $C_{(rs)(uv)}^{\quad(ab)}$ are continuous in the parameters d_{rs}, d_{uv} and d_{ab}, even though this is not manifest in the form (4.46), because the parameters r, s, etc. still enter in the ranges of the products.

Secondly, in principle one would have to carry out the $\Delta \to 0$ limit with the full expression (4.46). But the calculation can be significantly simplified if we use Dotsenko's observation [9] that the full structure constants can be obtained by analytic continuation from the $(r,1)$-subalgebra

$$C_{(r1)(u1)}^{\quad(a1)} = \prod_{m=1}^{k-1} \frac{\Gamma\big(\frac{m}{t}\big)\,\Gamma\big(\frac{1}{t}(m{-}d_{r1})\big)\,\Gamma\big(\frac{1}{t}(m{-}d_{u1})\big)\,\Gamma\big(\frac{1}{2t}(d_{r1}{+}d_{u1}{+}d_{a1}{+}1{+}t{-}2m)\big)}{\Gamma\big(1{-}\frac{m}{t}\big)\,\Gamma\big(\frac{1}{t}(d_{r1}{-}d_{m1})\big)\,\Gamma\big(\frac{1}{t}(d_{u1}{-}d_{m1})\big)\,\Gamma\big(\frac{1}{2t}({-}d_{r1}{-}d_{u1}{-}d_{a1}{-}1{+}t{+}2m)\big)}. \tag{4.49}$$

The idea is to treat d_{r1}, d_{u1}, d_{a1} as the fundamental variables and analytically continue in those. Again, this cannot be done as long as they appear (through k) in the range of the product. To remove this obstacle, rewrite $\Gamma(x) = \exp(\ln\Gamma(x))$ and use an integral formulation for $\ln\Gamma(x)$. The product in (4.49) then becomes a sum which can be carried out explicitly.

In more detail, we use that, for $x > 0$, [15, (8.341.6)]

$$\ln\Gamma\big(x\big) \; = I[\alpha\beta^x + x\beta + \gamma]\,, \tag{4.50}$$

where

$$I[f] = \int_0^1 \frac{\mathrm{d}\beta}{\beta(-\ln\beta)} f(\beta)\,, \qquad \alpha = \frac{1}{1-\beta}\,, \qquad \gamma = -\beta(\alpha{+}1)\,, \tag{4.51}$$

To carry out the sum one can use $\sum_{m=1}^{k} q^m = (q^k - q)/(q-1)$. The result is

$$\begin{aligned}\ln C_{(r1)(u1)}^{\quad(a1)} = I\Bigg[\frac{\alpha}{1-\beta^{1/t}} \Bigg\{ & \beta^{1/t}\big(1{+}\beta^{-d_{r1}/t}{+}\beta^{-d_{u1}/t}{+}\beta^{d_{a1}/t}\big) \;+ \\ & + \beta\big(1{+}\beta^{d_{r1}/t}{+}\beta^{d_{u1}/t}{+}\beta^{-d_{a1}/t}\big) \;- \\ & - \beta^{\frac{1+t}{2t}} \sum_{\varepsilon_r,\varepsilon_u,\varepsilon_a=\pm1} \beta^{\frac{1}{2t}(\varepsilon_r d_{r1}+\varepsilon_u d_{u1}+\varepsilon_a d_{a1})} \Bigg\} \;+ \\ & + \beta\cdot\frac{1-t}{t}\big(d_{r1}{+}d_{u1}{-}d_{a1}{-}1{+}t\big)\Bigg]\end{aligned} \tag{4.52}$$

where the integral converges for

$$-t < d_{r1}, d_{u1} < 1\,, \qquad -1 < d_{a1} < t\,, \qquad |d_{r1}| + |d_{u1}| + |d_{a1}| < 1 + t\,, \tag{4.53}$$

In the formulation (4.52) it is trivially possible to carry out the analytic continuation in d_{r1}, d_{u1}, d_{a1} by just substituting any set of real values in the ranges (4.53). Within this range, one can now verify (at least numerically) Dotsenko's result that the full structure constants (4.46) can be obtained by continuation from the subalgebra (4.49). The more precise statement is that (4.46) and (4.52) agree in the given range once they are both normalised to ${C_{kk}}^1 = 1$ through the use of (4.48).

We also immediately see how (4.52) turns into (2.6) when substituting $t = 1$.

5. Conclusion

We have considered the limit of the diagonal minimal models M_p as $p \to \infty$. The main aim was to find a bulk theory for the $c = 1$ boundary conditions considered in an earlier work [16], which were equally obtained in the $p \to \infty$ limit of minimal models. The expression for the cylinder partition functions with these boundary conditions implied that the bulk spectrum had to be a continuum. Thus the limit of the minimal model bulk theory was constructed in such a way that it yielded a continuous spectrum of bulk fields. Intriguingly, due to "hidden" analytic properties of the minimal model bulk structure constants the limit can be given explicitly.

The limit of expressions of minimal models led us to conjecture a non-rational CFT with $c = 1$ whose conformal boundary conditions include the set found in [16]. The main argument presented to support this conjecture was crossing symmetry of the two point function on the upper half plane and the four point function on the full complex plane. In the general case these were tested numerically, employing a recursive method due to Zamolodchikov to evaluate the conformal blocks at $c = 1$. In the case where all fields in the bulk four point function have half-integer labels the structure constants simplify considerably and additionally analytic expressions for the conformal four point blocks can be obtained. In the easiest case we presented an explicitly crossing symmetric expression for the bulk four point function.

There are however many remaining open questions. The first is the precise relation of the present theory to Liouville theory, to which it bears many superficial resemblances. Related to this point one may wonder if there is an action that describes the present theory and if in any way it can be understood as a quantisation of Liouville theory at $c = 1$. This will also help to find a physical interpretation of the M_∞ theory.

Rather than having to rely on numerical checks it would be interesting to see if the crossing of the $c = 1$ blocks can be described by fusion matrices, as was done in the case of Liouville theory in [21, 25]. The properties of the fusion matrices could then be used to prove crossing symmetry.

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A conformal field theory has to fulfill more consistency conditions than just the two instances of crossing symmetry tested here [24, 19]. Other constraints come from the four point function on the boundary, correlators involving two boundary and one bulk field, as well as one point functions on torus and cylinder. The first of these additional constraints was already conjectured to hold in [16]. To test the second, the coupling of the $c = 1$ bulk fields to arbitrary boundary fields (and not just the identity) has to be worked out. This would also allow one to verify the consistency of the limit of the bulk-boundary coupling constants with the decomposition of the boundary conditions at $c = 1$. We leave these problems for future research.

One may also wonder what the interpretation of M_∞ would be in the context of string theory, where one could replace a free boson describing a particular dimension in target space by M_∞. From this point of view it would be particularly interesting to repeat the calculation presented here in the case of $N = 1, 2$ super minimal models.

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A. Relation to Liouville structure constants

Here we compare in some detail the formula for the M_∞ structure constants (2.4) to the corresponding expression in Liouville theory, given by Dorn and Otto in [8] and by Zamolodchikov and Zamolodchikov in [30]. The following expressions are taken from [30].

Let $Q = b + 1/b$. The Liouville theory has central charge $c_L = 1 + 6Q^2$ and the exponential Liouville operators $V_\alpha(\zeta) = e^{2\alpha\phi(\zeta)}$ have conformal weight $\Delta_\alpha = \alpha(Q-\alpha)$. Define functions $\Upsilon(x)$ as

$$\log \Upsilon(x) = \int_0^\infty \frac{\mathrm{d}t}{t} \left[\left(\frac{Q}{2} - x \right)^2 e^{-t} - \frac{\sinh^2 \left(\frac{Q}{2} - x \right) \frac{t}{2}}{\sinh \frac{bt}{2} \sinh \frac{t}{2b}} \right] \tag{A.1}$$

The Liouville structure constants are then:

$$C(\alpha_1,\alpha_2,\alpha_3) = \left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{(Q-\alpha_1-\alpha_2-\alpha_3)/b} \cdot \frac{\mathrm{d}\Upsilon(x)}{\mathrm{d}x}\Big|_{x=0} \times \tag{A.2}$$
$$\times \frac{\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon(\alpha_1+\alpha_2-\alpha_3)\Upsilon(\alpha_2+\alpha_3-\alpha_1)\Upsilon(\alpha_3+\alpha_1-\alpha_2)}\,.$$

From the expression for the central charge $c_L = 1+6Q^2$ we see that for $c=1$ we need $Q=0$. To achieve this we set $b=i$. The conformal weights are then given by $\Delta_\alpha = -\alpha^2$. To bring them to the form $h_y = y^2/4$ used here let $\alpha = iy/2$.

As a next step some factors are dropped from (A.2) which can be absorbed by re-defining the fields. That is, we consider only the expression

$$\tilde{C} = \frac{\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon(\alpha_1+\alpha_2+\alpha_3)\Upsilon(\alpha_1+\alpha_2-\alpha_3)\Upsilon(\alpha_2+\alpha_3-\alpha_1)\Upsilon(\alpha_3+\alpha_1-\alpha_2)} = \exp\left(\int_0^\infty \frac{\mathrm{d}t}{t}F(t)\right), \tag{A.3}$$

To obtain the function $F(t)$ we substitute all expressions (A.1) and rewrite $\tilde{C}$ as the exponential of a single integral. In doing so we first notice that all terms x^2e^{-t} in the integrand of (A.1) cancel. For each term $\Upsilon(iy)$ in (A.3) we thus get a contribution

$$-\big(\sinh(-iyt/2)\big)^2\big(\sinh(it/2)\sinh(-it/2)\big)^{-1} = \frac{e^{it}}{(1-e^{it})}(e^{ity}+e^{it(-y)}-2) \tag{A.4}$$

to $F(t)$. If we now modify also the integration contour in (A.3) to run along the positive imaginary axis, i.e. we replace $t = i\tau$ where τ runs from 0 to ∞, the resulting expression is precisely the one given for $Q(x,y,z)$ in (2.6).

The correspondence between M_∞ and Liouville does however not seem to be completely straightforward. First note that to match $Q(x,y,z)$ to (A.3) we did indeed *replace* and not *rotate* the complex integration contour, that is we did not worry about poles. Second it is unclear how the step function $P(x,y,z)$ in (2.4) would appear from the Liouville point of view.

B. Analytic continuation of the $c=1$ structure constants

Recall the definition of $Q(x,y,z)$ in the bulk structure constants (2.4)

$$Q(x,y,z) = I\left[\frac{\beta}{(1-\beta)^2}\left\{2+\sum_{\varepsilon=\pm1}\left(\beta^{\varepsilon x}+\beta^{\varepsilon y}+\beta^{\varepsilon z}\right) - \sum_{\varepsilon_x,\varepsilon_y,\varepsilon_z=\pm1}\beta^{(\varepsilon_x x+\varepsilon_y y+\varepsilon_z z)/2}\right\}\right], \tag{B.1}$$

where the functional I is defined as

$$I[f(\beta)] = \int_0^1 \frac{\mathrm{d}\beta}{\beta(-\ln\beta)}f(\beta) = \int_0^\infty \frac{\mathrm{d}\tau}{\tau}f(e^{-\tau})\,; \qquad \beta = e^{-\tau}\,, \tag{B.2}$$

Here we will use the second formulation, and split the integral in two parts

$$I[f] = I_1[f] + I_2[f] \quad \text{where} \quad I_1[f] = \int_0^1 \frac{\mathrm{d}\tau}{\tau} f(e^{-\tau}), \;\; I_2[f] = \int_1^\infty \frac{\mathrm{d}\tau}{\tau} f(e^{-\tau}), \tag{B.3}$$

After splitting the integral in (B.1) in this way, one can verify that the part I_1 will converge irrespective of the values of x, y, z, while for I_2 to converge we need $|x|, |y|, |z| < 1$ and $|x + y + z| < 2$. Let

$$G(x) = I_2 \left[\frac{\beta}{(1-\beta)^2} \cdot \beta^x \right], \tag{B.4}$$

For $x > -1$ the integral converges and we can use (B.4) as definition for $G(x)$. For $x \le -1$, $G(x)$ can be defined by analytic continuation in x.

Before doing so, let us recall some properties of the exponential integral $E_1(x)$ and $Ei(x)$ [15, (8.211, 8.214)]

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t} \mathrm{d}t = -\gamma - \ln x - \sum_{n=1}^\infty \frac{(-x)^n}{n \cdot n!}\,; \qquad x > 0 \tag{B.5}$$

$$Ei(x) = -\lim_{\varepsilon \to 0+} \Big(\int_{-x}^{-\varepsilon} \frac{e^{-t}}{t} \mathrm{d}t + \int_\varepsilon^\infty \frac{e^{-t}}{t} \mathrm{d}t \Big) = \gamma + \ln x + \sum_{n=1}^\infty \frac{x^n}{n \cdot n!}\,; \qquad x > 0 \tag{B.6}$$

$$E_1(-x + i \cdot 0) = -Ei(x) - i\pi \tag{B.7}$$

Equation (B.7) describes the behaviour of $E_1(x)$ under analytic continuation along a path that passes above the origin.

Let us, for definiteness, fix the path along which we want to continue $G(x)$ to also pass just above the negative real axis and the origin.

Note that, for $x > 0$, $I_2[\beta^x] = E_1(x)$. One can quickly verify that the following functional relation holds for $G(x)$, $x > -1$:

$$G(x) = 2G(x{+}1) - G(x{+}2) + E_1(x{+}1)\,. \tag{B.8}$$

Upon analytic continuation the relation (B.8) will remain true, so that for $x \le -1$ we can use it to bring the argument of G into the positive domain. The analytic continuation of E_1 is just (B.7).

One can now solve this relation explicitly. Write $x < 0$ as $x = -n + f$ where $n \in \mathbb{Z}_{\ge 0}$ and $0 \le f < 1$. Then

$$G(x) = (n{+}1)\, G(f) - n\, G(f{+}1) + n\, E_1(f) - \sum_{k=1}^{n-1} (n{-}k)\, Ei(k{-}f) - \frac{n(n{-}1)}{2} i\pi \tag{B.9}$$

The function $G(x)$ thus has poles for negative integer values of x.

With the explicit form (B.9), the expression for $Q(x,y,z)$ which is valid for all values of $x,y,z \in \mathbb{S}$ is

$$Q(x,y,z) = I_1\left[\frac{\beta}{(1-\beta)^2}\left\{2+\sum_{\varepsilon=\pm1}\left(\beta^{\varepsilon x}+\beta^{\varepsilon y}+\beta^{\varepsilon z}\right)-\sum_{\varepsilon_x,\varepsilon_y,\varepsilon_z=\pm1}\beta^{(\varepsilon_x x+\varepsilon_y y+\varepsilon_z z)/2}\right\}\right]+ \\ +2G(0)+\sum_{\varepsilon=\pm1}\left(G(\varepsilon x)+G(\varepsilon y)+G(\varepsilon z)\right)-\sum_{\varepsilon_x,\varepsilon_y,\varepsilon_z=\pm1}G\left(\frac{(\varepsilon_x x+\varepsilon_y y+\varepsilon_z z)}{2}\right), \tag{B.10}$$

Recall that above we made an explicit choice for the path along which $G(x)$ was analytically continued. $G(x)$ is singular for negative integer values of x, and the path passed above these poles. Eq. (B.9) shows that the singularities are all logarithmic, so that a change of path can lead to an addition or subtraction of an integer multiple of $2\pi i$ in (B.9). Since in the end it is e^Q that matters, we see that the structure constants are independent of the path chosen for analytic continuation.

The imaginary part of $G(x)$, and thus that of $Q(x,y,z)$, is always of the from im, where $m \in \mathbb{Z}$. This implies that $c(x,y,z)$ as defined in (2.4) is always real. It could take any value in $\mathbb{R}$, though. However one can verify (at least numerically) that, with the choice of contour for $G(x)$ as in (B.9),

$$\operatorname{Im} Q(x,y,z) \neq 0 \quad \Rightarrow \quad P(x,y,z)=0\,, \tag{B.11}$$

In particular this implies that $c(x,y,z) \geq 0$.

C. Exact values of the $c=1$ structure constants

In the special case that x and y (say) are both half integer, the structure constants take the simple form

$$c\left(m+\frac{1}{2}, n+\frac{1}{2}, z\right) = 2^{-z^2-1}\, q_{m,n}(z)\,, \tag{C.1}$$

where $q_{m,n}(x)$ is a polynomial of degree $(m+m^3+n+n^3)$. We conjecture that this polynomial is

$$q_{m,n}(x) = \left[\prod_{k=-m-n+1}^{m+n-1}\frac{(k-x)^{m+n+1-|k|}}{(|k|+\epsilon)^{m+n+1-|k|-\epsilon}}\right]\times\left[\prod_{k=2-|m-n|}^{|m-n|-2}\left(\frac{k-x}{m+n+1-k}\right)^{|m-n|-|k|}\right], \tag{C.2}$$

where $\epsilon=0$ is $m+n$ is even and 1 if $m+n$ is odd, and in both products k runs over every other integer. We have checked this for many cases, but cannot prove that this is indeed the correct result. The various factors can be found by differentiating the expression (B.1) for Q with respect to z, in which case the integration with respect to β can be done to give an expression involving the digamma function ψ. For x and y half-integral this simplifies to a rational function of z, which can then be integrated to give the final form. The overall constant term however we have had to fix numerically.

D. Virasoro representations at $c = 1$

The commutation relations for the Virasoro algebra at central charge $c = 1$ are

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0}\frac{1}{12}(m-1)m(m+1)\,, \tag{D.1}$$

A highest weight state $|h\rangle$ is defined by the property

$$L_0|h\rangle = h|h\rangle\,, \qquad L_m|h\rangle = 0\,, \quad m > 0\,, \tag{D.2}$$

The Verma module $M(h)$ of weight h is generated by the action of the Virasoro algebra on a highest weight state $|h\rangle$ and has a basis of states

$$L_{-n_1}L_{-n_2}\dots L_{-n_k}|h\rangle \qquad \text{where} \quad n_1 \geq n_2 \geq \cdots \geq n_k \geq 1\,, \tag{D.3}$$

Hence, the character of $M(h)$ is

$$\chi_{M(h)}(q) = \mathrm{tr}_{M(h)}(q^{L_0-c/24}) = \frac{q^h}{\eta(q)}\,, \tag{D.4}$$

where $\eta(q) = q^{1/24}\prod_{n\in\mathbb{Z}_{\geq 0}}(1-q^n)$ is the Dedekind η-function.

The irreducible representation $L(h)$ of weight h is the quotient of $M(h)$ by its maximal proper submodule. The structure of Virasoro Verma modules is known [14], and in particular, for $c = 1$ a Verma module is irreducible unless $h = n^2/4$ for an integer n. If $h = n^2/4$ then the maximal submodule is generated by a single highest weight vector at level $|n|+1$ and is isomorphic to the Verma module $M(|n|+2)$, so that it is straightforward to write down the characters $\chi_x(q)$ of irreducible highest weight representations $L(h_x)$ with weight $h_x = x^2/4$:

$$\chi_x(q) = \begin{cases} \vartheta_x(q)\,; & x \notin \mathbb{Z} \\ \vartheta_x(q) - \vartheta_{|x|+2}(q)\,; & x \in \mathbb{Z} \end{cases} \quad \text{where } \vartheta_x(q) = \frac{q^{x^2/4}}{\eta(q)}\,, \tag{D.5}$$

We will also need the modular transformation properties of $\vartheta_x(q)$. Let $\mathrm{Im}\,\tau > 0$, $q = e^{2\pi i\tau}$ and $\tilde{q} = e^{-2\pi i/\tau}$. The modular transformation of the η-function is simply $\eta(\tilde{q}) = \sqrt{-i\tau}\,\eta(q)$. Using an inverse Laplace transformation[4] one can then check that

$$\vartheta_y(q) = \frac{1}{\sqrt{2}}\cdot\int_0^\infty (e^{\pi ixy} + e^{-\pi ixy})\cdot\vartheta_x(\tilde{q})\,\mathrm{d}x\,, \tag{D.6}$$

[4] More specifically [15, (17.13.114)]: $(\pi/s)^{1/2}e^{-a/s} = \int_0^\infty x^{-1/2}\cos(2\sqrt{ax})e^{-sx}\mathrm{d}x$ where $\mathrm{Re}(s) > 0$.

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In the main text this relation is used in the form

$$\int_0^\infty \mathrm{d}x\ 2^{3/2}\cdot \sin(\pi a x)\cdot \sin(\pi b x)\cdot \vartheta_x(\tilde{q}) = \vartheta_{a-b}(q) - \vartheta_{a+b}(q) = \sum_{k=|a-b|+1\,,\,2}^{a+b-1} \chi_{k-1}(q)\,, \tag{D.7}$$

where in the final sum k is increased in steps of two.

E. Zamolodchikov's recursive methods

In [26] and [28] Al. Zamolodchikov investigated the analytic structure of the conformal blocks as functions of the central charge c. This leads to the formulation of a recursion relation.

Consider a four point block with conformal weights h_1, h_2, h_3, h_4 and internal channel H, with the two insertions taken at the coordinates 1 and ζ

$$\begin{array}{c|c|c} h_1 & {}^{h_2}\ H & {}^{h_3}\ h_4 \\ \hline & 1 & \zeta \end{array} = \zeta^{H-h_3-h_4} F(c, h_i, H, \zeta)\,. \tag{E.1}$$

Here c is the central charge. In this definition of the function F the divergence for $\zeta \to 0$ has been extracted and F now has the expansion $F = 1 + a_1\zeta + a_2\zeta^2 + \ldots$. In order to work out the coefficient of ζ^n we have to sum over all intermediate states at level n. The function F then takes the form

$$F(c, h_i, H, \zeta) = 1 + \sum_{n=0}^{\infty} \frac{P_n(c, h_i, H)}{Q_n(c, H)} \cdot \zeta^n\,, \tag{E.2}$$

The factor P_n is a polynomial in c, h_i, H and originates from evaluating the three point functions for an insertion of states of the form $L_{-m_1}\ldots L_{-m_k}|H\rangle\langle H|L_{n_1}\ldots L_{n_\ell}$. The factor Q_n is a polynomial in c, H and results from computing the inverse of the inner product matrix of level n vectors in the highest weight representation $L(c, H)$. Consequently it is independent of the h_i.

For generic values of c and H, the highest weight representation $L(c, H)$ is just the Verma module $M(c, H)$. The determinant of its inner product matrix at level n is by definition the Kac-determinant (see [18] for more details).

If we understand $F(c, h_i, H, \zeta)$ as a function of c, equation (E.2) shows that its possible poles are located at zeros of $Q(c, H)$, i.e. at values of c where $M(c, H)$ becomes degenerate. For fixed H, the Kac determinant tells us that $V(c, H)$ has null vectors when ever c takes one of the values $c_{mn}(H)$, defined as follows

$$c_{mn}(H) = 13 - 6\big(t_{mn}(H) + (t_{mn}(H))^{-1}\big)\,, \tag{E.3}$$

where

$$t_{mn}(H) = \frac{2H + mn - 1 + \sqrt{4H(H + nm - 1) + (m - n)^2}}{n^2 - 1} > 0 \qquad \text{(E.4)}$$

and the indices m, n run over integer values $m \geq 1$ and $n \geq 2$. Since $H \geq 0$, the argument of the square root is always positive. Furthermore, we always choose the positive branch of the square root, i.e. $\sqrt{x} > 0$ for $x > 0$.

We now know the location of the poles of $F(c, h_i, H, \zeta)$ as a function of c. Zamolodchikov argues that these are simple poles and that the residues of the poles are again proportional to conformal blocks and gives the expansion

$$\begin{aligned} F(c, h_i, H, \zeta) =& {}_2F_1(H{+}h_1{-}h_2, H{+}h_3{-}h_4; 2H; \zeta) + \\ &+ \sum_{m\geq 1, n\geq 2}^{\infty} \frac{A_{mn}(H) P_{mn}(h_i, H)}{c - c_{mn}(H)} \cdot \zeta^{m\cdot n} \cdot F(c_{mn}(H), h_i, H{+}mn, \zeta)\,, \end{aligned} \qquad \text{(E.5)}$$

This is the sought after recursion relation. ${}_2F_1$ denotes the hypergeometric function. The factor $P_{mn}(h_i, H)$ is designed to give zero in case that the fusions $h_1 \times h_2 \to H$ and $h_3 \times h_4 \to H$ are allowed even at the degenerate point $c = c_{mn}(H)$. It is given by

$$P_{mn}(h_i, H) = \prod_{p,q} \Big\{ (\ell_2{+}\ell_1{-}\ell_{pq})\, (\ell_2{-}\ell_1{-}\ell_{pq})\, (\ell_3{+}\ell_4{-}\ell_{pq})\, (\ell_3{-}\ell_4{-}\ell_{pq}) \Big\} \qquad \text{(E.6)}$$

Here the product runs over $p = -m + 1, -m + 3, \ldots, m - 3, m - 1$ and $q = -n + 1, -n + 3, \ldots, n - 3, n - 1$. The ℓ_i and ℓ_{pq} are all functions of m, n and H and are defined as

$$\ell_i(m, n, H) = \sqrt{h_i + \frac{(1{-}t_{mn}(H))^2}{4t_{mn}(H)}}\,, \qquad \ell_{pq}(m, n, H) = \frac{p - q \cdot t_{mn}(H)}{\sqrt{4t_{mn}(H)}}\,, \qquad \text{(E.7)}$$

Up to this point the formulae where more or less determined by consideration of the analytic structure of the conformal blocks. The astonishing fact is that Zamolodchikov managed to find an explicit expression also for the normalisation factor $A_{mn}(H)$:

$$A_{mn}(H) = \frac{12\big(t - \frac{1}{t}\big)}{(m^2{-}1)\frac{1}{t} - (n^2{-}1)\, t} \cdot \prod_{a,b} \frac{1}{2\, \ell_{ab}}\,, \qquad \text{(E.8)}$$

Here it is again understood that $t = t_{mn}(H)$ and $\ell_{ab} = \ell_{ab}(m, n, H)$, as defined in (E.4) and (E.7). The product runs over the range $a = -m + 1, -m, \ldots, m - 1, m$ and $b = -n + 1, -n, \ldots, n - 1, n$. The two points $(a, b) = (0, 0)$ and $(a, b) = (m, n)$ have to be omitted from the product.

E.1 Recursion for internal channel $H = 0$

In the test of crossing symmetry we will also need the conformal blocks involving the degenerate internal channel $H = 0$. In this case the sum over states in the internal channel is taken over a smaller set, since the null vectors in the Verma module $M(c, H{=}0)$ have to be left out.

It turns out that the $H = 0$ block is not obtained by taking the limit $H \to 0$ of expression (E.1). Instead one has to slightly modify the recursion formula (E.5). Comparison with the explicit computation of the expansion in ζ suggests that only the first level of the recursion is affected. The relation we find is

$$\begin{array}{c|c|c} & h_2 & h_3 \\ h_1 & 0 & h_4 \\ \hline & 1 & \zeta \end{array} = \zeta^{-h_3-h_4} F_{\text{null}}(c, h_i, \zeta)\,, \tag{E.9}$$

$$F_{\text{null}}(c, h_i, \zeta) = 1 + \lim_{H\to 0} \sum_{\substack{m\geq 1, n\geq 2 \\ m\neq n}}^{\infty} \frac{A_{mn}(H) P_{mn}(h_i, H)}{c - c_{mn}(H)} \cdot \zeta^{m\cdot n} \cdot F\big(c_{mn}(H), h_i, H + mn, \zeta\big) \tag{E.10}$$

This relation of course makes sense only for $h_1 = h_2$ and $h_3 = h_4$ since otherwise $H = 0$ is forbidden. A_{mn}, P_{mn}, c_{mn} and F are the same as in the previous section. Note that in the first level of the recursion (i.e. in F_{null}) all diagonal terms $m = n$ are left out, while in higher levels (i.e. in F) they are again included. If one tries to start the recursion in (E.10) directly with $H = 0$, one finds that some of the A_{mn} become singular. In taking the $H \to 0$ limit these singularities cancel, and one obtains a finite answer.

E.2 Recursion relation from pole structure in H

The recursion relations (E.5) and (E.10), which we will refer to as c-recursion, have the advantage of directly producing the power series expansion in ζ. To check its correctness, this result can be compared term by term to the expansion resulting from inserting a basis of intermediate states in the corresponding block.

In [28] Zamolodchikov describes an alternative method to compute a conformal block, which derives from the pole structure in H rather than in c. This formulation of the blocks, which we refer to as H-recursion, has better convergence properties than the c-recursion, but it is less straightforward to extract the expansion in ζ.

To see if a program using these recursion relations works correctly, one could start by implementing and verifying the c-recursion, and then comparing it to the results from the H-recursion.

Below we reproduce the formulae for the H-recursion in the case of $c \leq 1$ and a non-degenerate internal channel H.

$$\frac{h_1 \left| \begin{matrix} h_2 \\ H \end{matrix} \right| \begin{matrix} h_3 \\ \end{matrix} h_4}{1 \quad \zeta} = \zeta^{H-h_3-h_4}\, \mathcal{F}(c,h_i,H,\zeta)\,,$$

$$\mathcal{F}(c,h_i,H,\zeta) = \frac{(16q)^{H+(1-c)/24} \cdot \mathcal{H}(c,h_i,H,q)}{x^{(1-c)/24+h_3+h_4}\,(1-x)^{(1-c)/24+h_2+h_3} \cdot \big(\theta_3(q)\big)^{(1-c)/2+4(h_1+h_2+h_3+h_4)}}$$

$$\mathcal{H}(c,h_i,H,q) = 1 + \sum_{m\geq 1,n\geq 1} \frac{(16q)^{m\cdot n} R_{mn}(c,h_i)}{H - H_{mn}(c)} \mathcal{H}(c,h_i,H_{mn}(c)+mn,q) \qquad \text{(E.11)}$$

Note that contrary to (E.5) and (E.10), in this recursion the sum starts at one for both m and n. The theta function is $\theta_3(q) = \sum_{n\in\mathbb{Z}} q^{n^2}$ and q is a function of ζ, given by

$$q(\zeta) = \exp\big(i\pi\tau(\zeta)\big)\,, \qquad \tau(\zeta) = i \cdot \frac{K(1-\zeta)}{K(\zeta)}\,, \qquad K(\zeta) = \frac{1}{2}\int_0^1 \frac{\mathrm{d}t}{\sqrt{t(1-t)(1-\zeta t)}}\,. \qquad \text{(E.12)}$$

The coefficients R_{mn} are

$$R_{mn}(c,h_i) = -\frac{1}{2}\,\frac{\prod_{p,q}(\lambda_2+\lambda_1-\frac{\lambda_{pq}}{2})\,(\lambda_2-\lambda_1-\frac{\lambda_{pq}}{2})\,(\lambda_3+\lambda_4-\frac{\lambda_{pq}}{2})\,(\lambda_3-\lambda_4-\frac{\lambda_{pq}}{2})}{\prod_{a,b}\lambda_{ab}}\,. \qquad \text{(E.13)}$$

The product in the numerator runs over

$$\begin{aligned} p &= -m+1, -m+3, \ldots, m-3, m-1\,, \\ q &= -n+1, -n+3, \ldots, n-3, n-1\,, \end{aligned} \qquad \text{(E.14)}$$

while the product in the denominator runs over

$$\begin{aligned} a &= -m+1, -m+2, \ldots, m-1, m\,, \\ b &= -n+1, -n+2, \ldots, n-1, n\,. \end{aligned} \qquad \text{(E.15)}$$

The two pairs $(a,b) = (0,0)$ and $(a,b) = (m,n)$ have to be omitted from the product in the denominator. λ_i and λ_{pq} are functions of c given by

$$\lambda_i(c) = \sqrt{h_i + \frac{1-c}{24}}\,, \quad \lambda_{pq}(c) = \alpha_+ \cdot p + \alpha_- \cdot q\,, \quad \alpha_\pm = \frac{1}{\sqrt{24}}\Big(\sqrt{1-c} \pm \sqrt{25-c}\Big)\,, \qquad \text{(E.16)}$$

Finally

$$H_{mn}(c) = \frac{c-1}{24} + \frac{\big(m\cdot\alpha_+(c) + n\cdot\alpha_-(c)\big)^2}{4}\,, \qquad \text{(E.17)}$$

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